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VOLUME I

A Publication of the

INDUSTRIAL MATHEMATICS SOCIETY
5135 CASS AVENUE, DETROIT 2, MICHIGAN

FOREWORD

The Industrial Mathematics Society was formed as a forum for discussion of matters intermediate between the advanced theory called applied mathematics and the routine formulae found in engineering handbooks. The broad interest in this intermediate field, not only among engineers but also among mathematicians has been a source of surprise and gratification to the Society.

In view of this wide interest it seemed desirable to make a permanent record of the papers presented before the various meetings of the Society. While there is some hesitancy in starting still another scientific journal, it appears that there is need for a publication in the intermediate field mentioned above.

This first issue is a compilation of papers presented during the Society's first year and it is intended to continue on an annual basis, at least until the quantity of papers presented before the Society forces more frequent publication.

Orientation Lecture Series

"ELEMENTARY DIFFERENTIAL EQUATIONS OF FLUID MECHANICS, DYNAMICS AND STRENGTH OF MATERIALS"

by

Dr. Paul F. Chenea
Department of Engineering Mechanics
University of Michigan

Presented at a meeting of the
Industrial Mathematics Society
On Friday, September 23, 1949

THE BASIC PRINCIPLES OF CLASSICAL MECHANICS

The basic principles of Classical Mechanics may be divided into four categories. These categories and the corresponding basic principles are as follows:

I. The Natural Laws, which are universal in character. These laws have been developed gradually since man first concerned himself with that branch of science now known as Mechanics. There are four such laws:

A. The Law of Conservation of Mass states that mass can neither be created nor destroyed. Modern physics takes exception to this point of view for those processes involving atomic fission, however this development is ruled out of Classical Mechanics. The Law of Conservation of Mass may be stated mathematically as

$$\frac{d}{dt}(M) = 0$$

where (M) is the total mass contained in a closed system and (t) is time.

B. The Laws of Motion are associated with the great scientist Sir Isaac Newton. Modern physics also takes exception to this basic principle in the form stated below in that it makes suitable modifications. These modifications become important only when the velocities of the bodies approach the velocity of light, a phenomenon which rarely occurs in Classical Mechanics. The Laws of Motion may be stated in vector form as

$$\frac{d}{dt}(\overrightarrow{Mv}) = \vec{F}$$

where $\vec{v}$ is the velocity vector and (F) is the resultant force vector acting on the mass (M).

C. The Law of Conservation of Energy is frequently called the generalized First Law of Thermodynamics. This law, which is also invalidated by atomic fission, may be stated mathematically as

$$\frac{d}{dt}(\varepsilon) = 0$$

where (ε) is the total energy in a closed system. The total energy may be made up of kinetic energy, heat, potential engy, etc.

D. The Law of Inadmissibility of Decreasing Entropy is also called the Second Law of Thermodynamics. This law may be stated symbolically as

$$\frac{d}{dt}(S) \geq 0$$

where (S) is the total entropy in a closed system. The usefulness of this law is somewhat limited in those instances in which the entropy is not constant. Since this is a common situation in nature, this fundamental principle sometimes makes it difficult to draw conclusions from our mathematical analyses.

II. Kinematic Relations fall into two broad classes, namely, definitions and continuity conditions.

A. The definitions needed to formulate the whole of Classical Mechanics in a concise fashion are almost innumerable. A typical example of these is the definition of strain

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

where (ε_x) is the elongation per unit length in the (x) direction, called strain, and (u) is the displacement in the (x) direction.

B. The continuity conditions insure that the masses being studied do not perform in a manner prohibited by the design of the system. Examples of this type of conditions are the well-known compatibility equations of the theory of elasticity.

III. Equations of State relate the variables which describe a static state of the body to each other. They do not involve time. Examples of these equations are:

The Ideal Gas Law:

$$\frac{P}{\rho} = RT$$

in which (P) is the pressure, (ρ) is the mass density, (T) is the temperature, and (R) is a constant.

Hooke's Law for an elastic rod

$$\varepsilon = \frac{\sigma}{E} + \alpha T$$

where (ε) is the strain, (σ) the stress, (T) is the temperature and (E) and (α) are the constants.

IV. The Dissipation Relations describe the manner in which energy is transformed. These relations involve time and/or time rates of change. Two examples of these relations are:

Newton's Law of viscositv for a fluid:

$$\tau_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\mu$$

where (τxy) is the shear stress in the (xy) plane, (u) and (v) are the velocities in the (x) and (y) directions, respectively, and (μ) is a constant.

Fourier's Law of heat conduction:

$$q_x = -K\frac{\partial T}{\partial x}$$

where (qx) is the rate of heat flow per unit area in the (x) direction, (T) is the temperature and (K) is a constant.

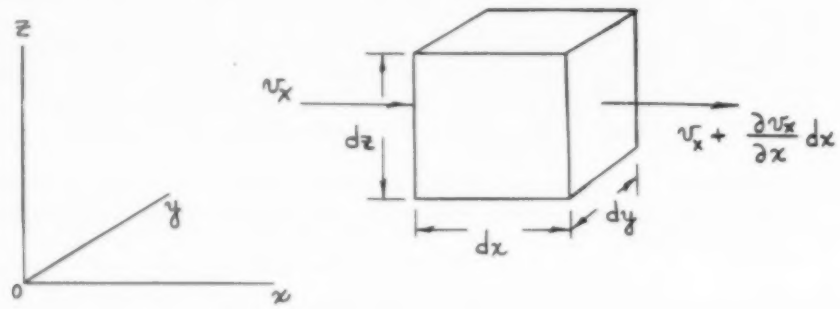
It should be noted only the natural laws are of universal application and that the form of the remaining basic principles depend upon the problem and material at hand. It is quite apparent that all of these principles with the exception of equations of state involve space and time derivatives. For this reason, most of the fundamental equations of mechanics are differential in character.

A few examples, illustrating the principles stated above and a standard approach for continuous bodies are considered below. The method utilized, namely that of considering an infinitesimal element of the body of the form of a rectangular parallelepiped is not the most elegant procedure that can be devised, but it has great simplicity.

1. Example of the Law of Conservation of Mass:

THE CONTINUITY EQUATION FOR FLUID FLOW

Consider an infinitesimal volume element in the body as shown.



The Law of Conservation of Mass requires that

"increase in mass content" = "net inward mass flow"

Then, if (ρ) is the density and (v_x), (v_y) and (v_z) the velocities in the three coordinate directions, we have, per unit time:

$$\frac{d\rho}{dt} dx\, dy\, dz = \rho v_x\, dy\, dz - \rho\left(v_x + \frac{\partial v_x}{\partial x} dx\right) dy\, dz$$

$$+ \rho v_y\, dz\, dx - \rho\left(v_y + \frac{\partial v_y}{\partial y} dy\right) dz\, dx$$

$$+ \rho v_z\, dx\, dy - \rho\left(v_z + \frac{\partial v_z}{\partial z} dz\right) dx\, dy$$

or

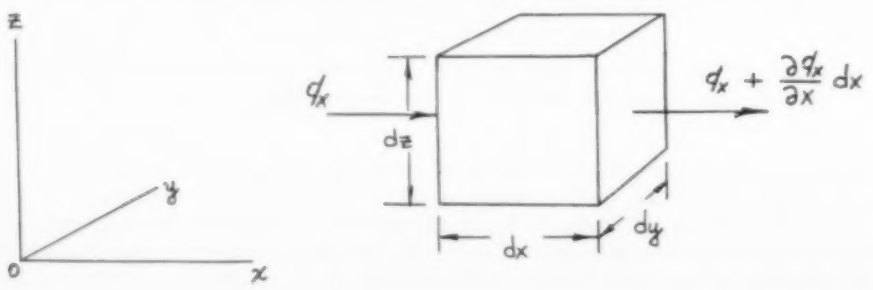
$$\frac{d\rho}{dt} = -\rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$

where ($\frac{d\rho}{dt}$) is the total rate of change of the density. This is the usual form of the continuity equation.

2. Example of the Law of Conservation of Energy:

THE HEAT CONDUCTION EQUATION

Consider an infinitesimal volume element in the body as shown.



The Law of Conservation of Energy requires that

"increase in energy content" = "net inward energy flow"

Then, if (c) is the specific heat, (ρ) is the mass density, (T) is the temperature and (q_x), (q_y) and (q_z) are the rates of heat flow in the three coordinate directions, we have, per unit time:

$$c\rho \frac{\partial T}{\partial t} dx\, dy\, dz = q_x dy\, dz - \left(q_x + \frac{\partial q_x}{\partial x} dx\right) dy\, dz$$

$$+ q_y dz\, dx - \left(q_y + \frac{\partial q_y}{\partial y} dy\right) dz\, dx$$

$$+ q_z dx\, dy - \left(q_z + \frac{\partial q_z}{\partial z} dz\right) dx\, dy$$

or

$$c\rho \frac{\partial T}{\partial t} = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right)$$

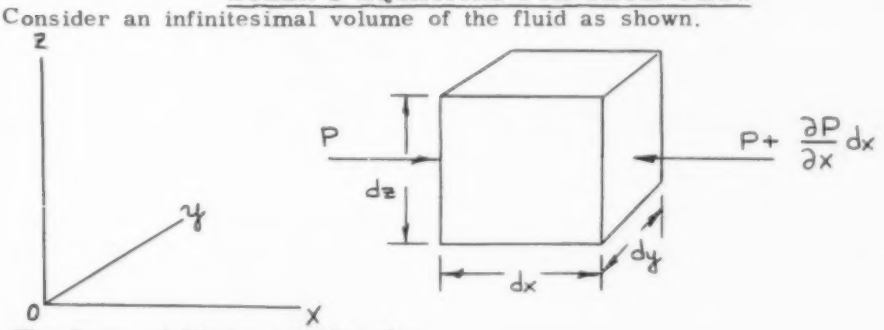
It is common to introduce Fourier's Law of heat conduction into this equation and thus obtain the usual form:

$$\frac{\partial T}{\partial t} = \frac{K}{c\rho}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

3. Example of the Laws of Motion:

EULER'S EQUATIONS FOR FLUID FLOW

Consider an infinitesimal volume of the fluid as shown.



The Laws of Motion require that

"rate of change of momentum" = "resultant force"

$$\frac{d}{dt}(\rho v_x\, dx\, dy\, dz) = P\, dy\, dz - \left(P + \frac{\partial P}{\partial x} dx\right) dy\, dz + \rho X\, dx\, dy\, dz$$

where ($\rho dx dy dz$) == the mass of the element, (P) is the pressure and (X) is the body force in the (x) direction per unit mass. Thus, for each direction, there is one equation of the form

$$\rho \frac{dv_x}{dt} = -\frac{\partial P}{\partial x} + \rho X$$

$$\rho \frac{dv_y}{dt} = -\frac{\partial P}{\partial y} + \rho Y$$

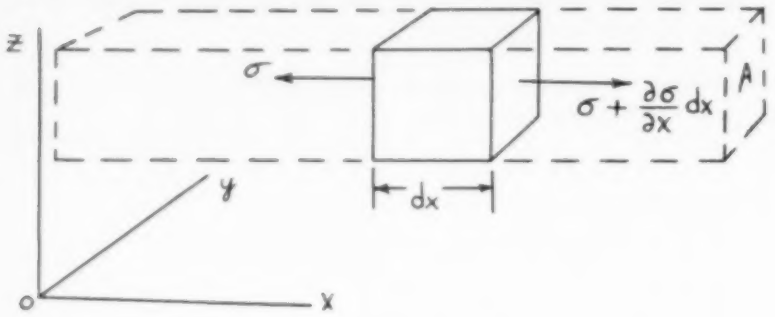
$$\rho \frac{dv_z}{dt} = -\frac{\partial P}{\partial z} + \rho Z$$

where ($\frac{dv}{dt}$) is the total time rate of change of the velocity.

4. Example of the Laws of Motion

LONGITUDINAL VIBRATIONS OF AN ELASTIC BAR

Consider an infinitesimal slice of an elastic bar as shown.



The Laws of Motion require that

"rate of change of momentum" = "resultant force"

$$\frac{\partial}{\partial t}(\rho A v\, dx) = -\sigma A + \left(\sigma + \frac{\partial \sigma}{\partial x} dx\right) A$$

where (ρ) and (A) are nearly constant for small motions, and (σ) is the tensile stress in the bar. Thus

$$\rho \frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial x}$$

It is common to introduce the definition of velocity and strain as well as Hooke's Law

$$v = \frac{\partial u}{\partial t} \qquad \varepsilon = \frac{\partial u}{\partial x} \qquad \sigma = E\varepsilon$$

where (u) is the displacement in the (x) direction. The basic equation may now be written as

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2}$$

which is the usual form of this equation.

APPLIED MATHEMATICS IN THE ELECTRIC POWER INDUSTRY

J. E. Freer

With the growth of a complex industrial system there must be a decided change from the use of engineering of the cut-and-try methods of the past. Engineers have to deal with more complicated and more interconnected physical phenomena than physicists or mathematicians, since, as a whole, idealized conditions do not exist in the physical world. As the complexity of engineering increases it becomes necessary to organize the solution of such problems.

Fortunately there are only a few fundamental elements which form the great variety of complicated engineering structures. Also, electrical and mechanical equations are identical mathematically, and consequently the two types of problems may be handled in the same way.

This evening I would like to talk about the problems of an electrical system and their solution. Stated simply, the problem of a power company is to deliver electric power to your lamp or motor at a fixed voltage and frequency at the instant you request it by turning on a switch. The power company doesn't know when or how much you are going to require, and the electricity is stored in coal piles or in water behind a dam. However, through the study of daily load curves, it is possible to predict the approximate amount of generating capacity it is necessary to have ready for use at any hour of the day and to predict when it is necessary to buy new generators for yearly load growth.

Between generators and loads, there is a transmission system which must grow with load and generation. In order to give reliable, uninterrupted service to a customer it is necessary to provide several routes over which the electricity may travel in order that interruption of one or more of the transmission lines due to lightning, sleet or insulation failure will not cause loss of service.

Another factor mentioned previously is that the proper voltage must be maintained at various points. Since the amount of voltage loss in a transmission line is directly proportional to the amount of current in the line, the voltage at different times of the day will vary unless there are corrective measures taken. New lines which are added to the system cannot be added on a haphazard basis, since high voltage lines are expensive to construct. The new line must carry a reasonable amount of current in order to be economical.

From this short summary it may be seen that it is necessary to calculate what will happen when a new piece of equipment is to be added to the system. Let's see what is involved in the calculation.

Steady State Solutions

As I mentioned previously, there are only a relatively few elements in a power system. These are resistance, reactance, voltage sources, and transformers. Every piece of equipment in a power system can be represented by combinations of these basic elements.

Paper presented at a meeting of the Industrial Mathematics Society on February 28, 1949. J. E. Freer is with the Detroit Edison Company.

Having a connection diagram of the system, it is possible to draw an impedance diagram showing the constants of the electrical system to be studied and the manner of interconnection. Knowing the magnitude of the loads and the amount of generation available, it would be possible to calculate the performance of the electrical system.

Calculation is highly impractical, however, because of the number of equations involved. A modern metropolitan electrical system problem would require, with simplifications and combinations of equipment, perhaps 200 simultaneous complex number equations with as many unknowns. In addition it would be desirable to have from 50 to 75 different combinations of the same problem with different parameters.

When adding new equipment to an electrical system, costs running into millions of dollars make it necessary to have solutions of these problems and the electrical industry has been foremost in the use of large-scale electrical calculators. By the use of the calculators, problems which would take a year to solve mathematically may be done in an hour or two, thus relieving the engineer of long involved calculations and enabling him to concentrate on finding the best solution.

There are three types of problems to be solved in electrical system work. The most common is a load and voltage study. The others are stability and short circuit calculations. The electrical calculator used in these studies is called a network analyzer, or network calculator, and consists of an assembly of adjustable resistors, reactors, capacitors, a number of sources of a.c. voltage, adjustable both in phase and magnitude. Provision is made for connecting these units together to form a network and instruments are used for measuring scalar and vector values of voltage, current, and power anywhere in the network. The instruments are designed in such a way that their insertion into the network does not appreciably change conditions. Since the calculator is essentially a miniature representation of a large power network it becomes necessary to work with only a small fraction of the voltage used in the real system where voltages up to 500,000 volts are found. One hundred volts is common voltage used in calculators. The current is quite low compared with the original current, causing conventional instruments to have too large a burden in comparison with the quantity to be measured, making necessary the use of vacuum tube amplifiers.

Transients

The alternating current calculator is essentially a device for solving steady state simultaneous equations of a complex variable. It is necessary, however, to study an electrical system during a fault condition or the sudden application of a load or interruption of a transmission line which is heavily loaded. Such sudden changes must be studied by means of differential equations. Again the mass of equations can be calculated but the solution is much easier on the network calculator. These types of studies are known as transient stability studies. Since a stability problem starts out the same as a load and voltage problem, perhaps showing the factors involved would bring out the full capabilities of the network calculator.

Power system stability is the term applied to an alternating current electric power system, denoting a condition in which the various synchronous machines of the system remain in step; conversely, instability denotes a condition involving falling out of step.

Consider a very simple system composed of a synchronous generator having no internal impedance connected through a transmission line having a series inductive reactance to a synchronous motor having no internal impedance. The voltage on the generator is an alternating voltage changing from zero to maximum at a constant rate. Taking a picture of the voltage at some instant we get a vector stationary in space and we will use this instantaneous voltage in our illustration. If the motor has a load, it

can be shown that the voltage on the motor is a vector displaced from the generator voltage by an angle which is dependent upon the amount of load and, since the motor is a synchronous machine, the rotor of the motor has the same angular displacement from the generator rotor as the angle between the voltages. The power transmitted from the generator to the motor then is dependent upon the angle between the two voltages. If we were to plot the power transmitted against the angle between the two voltages, we would have a sine function and we would find that the maximum power would be transmitted when the angle is 90°. Beyond that point the power falls off as does the sine function.

Suppose that we have the motor loaded such that the angle is 90° and we then add a little more load to the motor. The shaft of the motor is slowed down and the angle between the rotors of the two machines increases. Normally the power input would increase, but beyond the maximum point the power decreases, and the load will further retard the rotor so that the motor falls out of step and probably stalls. The point of maximum power transferal is the steady state stability limit of the system.

If a large load is suddenly applied to the motor when it is below the steady state stability limit, it may fall out of step even though the load does not exceed the steady state stability limit. The upper limit of the suddenly applied load is known as the transient stability limit. The transient stability limit is always below the steady state limit, but may have different values depending on the nature of the disturbance.

The most severe type of disturbance to which a power system is subjected is a short circuit and, therefore, the effect of short circuits must be determined in a stability study.

A three-phase short circuit on the pure reactance of the circuit connecting motor and generator entirely cuts off the flow of power between the machines. The generator output becomes zero along with the input to the motor. Because of the slowness of action on the governor of the prime mover driving the generator, the mechanical input to the generator remains constant for perhaps three-quarters of a second. As the power from the generator is decreased and its input remains constant, the generator speeds up and, unless the short circuit is removed rapidly, the generator will fall out of synchronism with the rest of the system and will have to be disconnected. In the study of an electrical system then the stability of the system must be taken into account in order to provide sufficient generation.

In a stability problem we want to study an interval of perhaps three-quarters of a minute. First we set up the network to scale on the calculator, the generators being represented by a reactance in series with a power source. Next the phase and magnitudes of the several power sources and the load impedances are adjusted to represent normal operating conditions. A short circuit is then applied at some point and the voltages of the power sources are readjusted to their value before the fault occurred. Then the power output of each machine is read by means of a wattmeter. A difference between the reading before and after the fault represents power available for acceleration or de-acceleration of the machine, and the amount is computed giving an angular change in the position of the machine in a small time interval. The angular change is made on the calculator and a new set of readings taken for the next time interval. These angular positions are plotted against time. At the appropriate time for a circuit breaker opening to clear the fault, the change is made in the calculator and the study progresses until it is apparent that the machine will remain in synchronism.

It will be seen then that the network calculator may be used to solve differential equations as well as steady state equations. As I mentioned previously, there are only a few basic elements to form a great variety of engineering structures. The network calculator was designed primarily for studying electrical system problems, but

its use has been extended to the study of involved mechanical problems. It makes possible the computation of problems which were impossible because of the length of time necessary to do them.

DESIGN AND CONSTRUCTION OF DIGITAL COMPUTING MACHINES

Carl C. Chambers

Introduction

The development of large scale digital computers in recent years is of epoch-making importance. Computing devices are not new, having been in use since before the birth of Christ. However, until very recently, the limitations of their capacity to compute has been so great that mathematicians devoted the bulk of their time to the development of logic to avoid computation and engineers approximated their analyses in such a way that the necessary computation would be reduced to a minimum. The work involved the devising of various methods of converting problems of a complex nature into simplified forms to make maximum use of the available tables of certain laboriously computed functions, such as Bessel's functions, sines and cosines, etc. Sometimes this resulted in a very accurate analysis but, more often, the approximations required were so radical that the low accuracy attained made the results of little value. Large scale, high speed computers throw a different light upon this attack. Problems become approached for solution in terms of their fundamental algebraic laws or, in machine terminology, their orders. It is this very fundamental difference in applied mathematics which is so important. Here mathematical manipulations must be restudied even to the extent of such an elemental process as recognizing equality.

Here is the starting point of the design of digital computers. In view of the machine complexity and the time required to carry out elemental operations, which operations should be considered most desirable as the orders for the machine? As an example, consider again the problem of recognizing the equality of two numbers. Should we look at each digit of one number in comparison with the corresponding digit of the other number? Should we on the other hand compare their magnitudes - is the first larger than the second, then is the second larger than the first? Depending upon the speed and the required equipment one of these processes should be chosen. But look again. If we follow the second procedure and the numbers are not equal, we have at hand other information which may be useful. And so we go on. Even the simple operation of identifying equality is found to require the close coordination of equipment design and mathematical logic of a surprisingly elemental nature.

In my discussion here, I will limit myself to the large-scale high speed computers which lead to this change in mathematical methods. Within this field, I want to emphasize two important considerations in planning and conducting a design and construction program for such a computer.

First, consider some unusual problems in estimating a priori the progress of the work. There is a definite tendency to encounter dreamers who disdain and refuse to recognize, or at least fail to anticipate, practical difficulties in a large system engineering job. Their tremendous enthusiasm and vision leads to extremely optimistic estimates for future accomplishments. This type of enthusiasm makes development blossom but makes administration difficult.

Paper presented at a meeting of the Industrial Mathematics Society on May 25, 1950.
Dr. Chambers is acting Dean of the Moore School of Electrical Engineering, University of Pennsylvania.

Second, consider the importance and meaning of reliability of a different order than that of the previous experience of electronic circuit designers. Errors in communications of the order of one in 100,000 bits of information would represent very high quality sound reproduction, while one error in 10^{12} bits of information in the EDVAC would make it unsatisfactory for problems of a few hours duration. An examination of the time to repair the machine when needed and to repeat the processing of the problem shows some very interesting requirements on the reliability of computers. Even with perfect checking systems, very few failures which require repair reduce the useful operating time to a negligible proportion of the total time.

My experience through association with the trials and tribulations of the construction of two such machines leads me to this selection for my "down-to-earth" discussion today.

Predicting and Planning the Program

The engineers and scientists who work in this field have a more than normal imagination, logic, and broad scientific background. They are familiar with electronics and are well trained in truly basic mathematics. This mathematical base not only includes the usual studies in algebra and calculus, but also a thorough understanding of mathematical philosphy of the type known by the high-sounding name of symbolic logic. Generally, they have encountered in their past experience some extremely complex analytical problems in some scientific field wherein a giant calculating machine would have made possible an otherwise impossible computation.

It is little wonder that such men are not at the same time practiced in the art of predicting accomplishments in a construction program. They can hardly be expected to have the characteristics of a successful factory manager or the "feel" for the multitude of indirect problems of the telephone systems engineer. They will not allow in their estimates for such cases as the long-term solvent action of the cement on the insulating tape on the wire enamel causing short circuits of the transformers.

The evidence of this imagination on the part of many in this field is clear as we note the newspaper reports of predictions in this field. Little more grandiose computers than those announced in the papers produce stories for the scientific pulp magazines. In fact, I understand that some computer scientists are actually contributors to these magazines. To predict that through these machines, everyone may have eleven-month vacations each year is not surprising.

Despite my ridicule of these predictions, I am convinced that digital computers and the techniques of their design and application will make a tremendous difference in the life on this planet. Further, I appreciate the advantages of startling pronouncements and predictions toward the promotion of work of this nature.

However, a distinct separation must be recognized between promotional predictions and the more down-to-earth planning predictions. Some of the predictions for the construction of machines to do amazing things for us in the next five years are like saying that combination autos and flying machines will be in the hands of five million people in 1955. Perhaps it is possible but there are a lot of problems to be solved before that time. Such a statement might well sell stock, but the SEC wouldn't like it. Many of the electronic computer scientists as promoters rival the Florida real estate promoters of a couple of decades ago. But despite this Miami is a beautiful vacation spot, and likewise we may expect great things from the electronic computers. However, let's not overlook the hurdles to be crossed so that this development does not pass through bankruptcy.

How can we protect against this calamity? Promote if you must, but plan on the basis of detailed studies, not on promotion. Let the experienced construction man plan the program. Force the computer scientist to develop all details before design and construction, or the latter will be a true gamble.

During the progress of the work beware of new proposals. Changes toward improvement, or more dangerously, to extend the usefulness of the computer generally set back the overall program. It is difficult to avoid being persuaded to add a shifting register which would only add one panel of 50 tubes and can be placed in the space at the top of the input cabinet so that aerodynamic problems for ultrasonic speeds can be solved in one-third of the time otherwise required. For example, our EDVAC was originally planned to have 1,200 tubes. In 1947 it was up to about 4,000 tubes. Through a change to crystal diode gating circuits it dropped to 2,000 tubes and 5,000 crystal diodes. By early 1948 it was almost 4,000 tubes and 8,000 crystal diodes. True, the final EDVAC was far better than that planned with 1,200 tubes, but a less satisfactory computer available for problems much earlier might well have been more valuable.

Let us build some computers but refuse to expect a machine under construction to keep abreast with the currently new proposals in the field.

Reliability of Digital Computers

In the EDVAC, information, be it number or instruction, is a succession of go or no go conditions, the presence or absence of pulses. In many of these circuits pulses, or the absence of pulses, are passed once every microsecond. Should a tube or a crystal rectifier miss a beat, an error is almost certain to occur. Thus, with the number of such elements in the EDVAC one missed pulse in 10^{12} would cause an error about once an hour. A probability of error of one ten billionth of one per cent would mean that problems of an hour duration would be in trouble.

To transmit audible sounds of extremely high quality requires about 100,000 yeas or nayes per second. An error of one of these would correspond to the loss of one-half cycle of the 10,000 cycle component per second - about the equivalent of one grain of dust on a very high quality phonograph record.

Thus the reliability of computers must be some 10 million times as good as an excellent quality audio amplifier. Presented in this way, computer reliability appears to be a staggering problem. However, the situation in fact is quite less severe.

Let us consider first the random errors - those errors where a pulse is lost or gained by chance due to a truly unordered random current such as that of the pitter-patter of electrons striking the plate of a vacuum tube. Here the probability of an interfering pulse being large enough to compare with the information varies as an exponential function of the amplitude. Because of this exponential character of the probability, we can say - loosely to be sure - that it is twice as hard to get a probability of error of one in 10^{12} as to get a probability of one in 10^{6}. Even at the present time, the probability of such errors in the EDVAC is less than one in 10^{12}. Were we to do half again as good - exponentially speaking - we would have a probable error of less than one in one hundred years.

A more careful analysis of this type of error indicates that such an expectation is very reasonable. However, relaxing our effort but slightly may cause an unbearable number of errors to be encountered. In many places, as the components approach failure, the random errors increase very rapidly. Some system of preventative maintenance is highly essential.

Errors due to failure of the circuitry are almost always characterized by continued or ordered errors. Here even the simplest checking systems will avoid using erroneous results. I have said almost always. The bad cases are those where a failure produces systematic but hidden errors. For instance, certain components of the machine function only in the case of the least significant digit; a failure might cause a consistent but hidden error. I believe that much can be done in the logic and design of the computer to avoid such possibilities. (This consideration generally indicates an advantage of the serial type machine in contrast to the parallel type machine).

The need for automatic checking systems has been the subject of many discussions. However, these have been largely discussions and studies of systems of checking. I want to raise the question of the advisability of any elaborate checking system. Checking systems which are intended to find random errors due to truly unordered effects should be made unnecessary by somewhat more conservative design, together with a preventative maintenance program. Simple checking - perhaps simply a check problem - is needed to locate part failures. Part failures causing hidden errors should be attacked by considered design.

Preventative maintenance directed toward keeping the probability of random errors safely below the danger point is difficult to accomplish. Perhaps the periodic replacement of all components, such as tubes, which critically affect this probability would be sufficient. An interesting proposal to this end is to decrease the voltages or other parameters of the design which control the safety factor against possible random error sources.

Checking systems generally require considerable equipment which may itself be a cause of trouble. Then again a checking system is useful only if it finds errors. If it finds very many such errors, the useful computing time will approach zero. These checking systems are valuable if they find a few errors but none of the machines is useful if the checking system were to find many errors.

Summary

Digital computers are wonderful machines, particularly with respect to their long-term potentiality. There is considerable danger of over-promotion in this field. Planning a program should be done on a detailed basis and not on the basis of the promotional pronouncements.

Reliability is a problem of an order new to electronic designers. However, equipments to date indicate that the required reliability is not at all unreasonable.

Recent contentions that elaborate checking systems are essential should be carefully examined since they are of no use if they find no errors, and the computer as a whole is a failure if they find many errors.

"ANALOGIES IN FLUID FLOW, ELASTICITY AND SOAP FILMS"

Dr. G. E. Hay[1]

1. Introduction. In this paper it is proposed to present some of the fundamentals of the mathematical theories of elasticity, hydrodynamics, and soap films. Each of these three theories is quite extensive, and can easily constitute the subject matter of a large number of papers such as the present one. Accordingly only enough of the fundamentals of these three theories will be presented here in order to demonstrate an analogy between them. As we shall see, this analogy exists because three problems, one from each of the three fields mentioned above, all reduce to the same problem in mathematics. These three problems are the following: (a) the torsion of a beam of uniform cross-section; (b) the two-dimensional flow of an incompressible perfect fluid inside a hollow rotating cylinder; (c) the equilibrium of a soap film stretched across the open end of a tube, the pressure inside the tube being greater than that outside the tube.

The importance of the analogy under consideration here lies in the fact that it is possible to obtain information about the solutions of any two of the above three problems through experimental work carried out on the third problem.

ELASTICITY

2. A Study of Stress. A body is said to be elastic if it suffers no permanent distortion when a load is applied to it and is then removed. This property of being elastic then depends both on the nature of the body and on the magnitude of the load. In this paper we shall consider only elastic bodies.

Let us consider a general elastic body, such as the one occupying the region (R) in Figure 1. We shall neglect effects due to gravity. In order to describe the behavior of this body mathematically, we introduce rectangular cartesian coordinates (x_1), (x_2), (x_3), as shown. Throughout this talk we shall adopt the convention that Latin subscripts range over the values 1, 2, 3. Accordingly we can denote the three rectangular cartesian coordinates collectively by the single symbol (x_i).

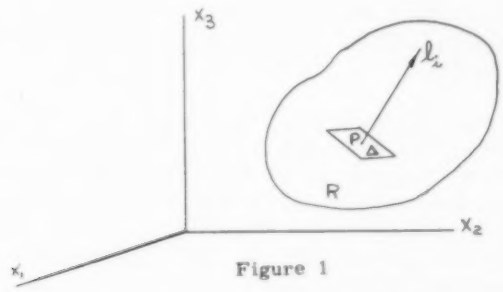


Figure 1

We shall now introduce functions which describe the internal reactions between all pairs of adjacent particles in the body. Let (P) be a general particle in the body, with coordinates (x_i). Let (l_i) be the components of a unit vector at (P), and let (Δ) be the plane through (P) and perpendicular to the unit vector (l_i). The stress

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[1] Dr. G. E. Hay is in the Mathematics Department of the University of Michigan. This paper was presented at a meeting of the Industrial Mathematics Society on Friday, November 4, 1949.

vector (T_i) can now be defined. It is the force per unit area at (P) exerted across (Δ), by those particles which lie on the same side of (Δ) as the vector (l_i), on those particles on the other side of (Δ). Obviously (T_i) are three functions of the coordinates (x_i) of (P) and the components of the unit vector (l_i) normal to (Δ). To denote this we can write

$$(2.1) \qquad T_i = T_i(x, l).$$

We shall now introduce the stress components. There are nine of them, which we denote by the symbol (T_{ij}). The three components (T_{ij}) are the components of the stress vector at (P) when (l_i) is made to point in the direction of the positive (x_1) axis. The definitions of (T_{2j}) and (T_{3j}) are similar.

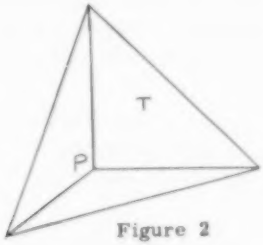

Figure 2

Let us now consider the tetrahedron (T) shown in Figure 2. One corner of (T) is at the point (P), and the three faces through (P) are parallel to the coordinate planes. The fourth face is parallel to the plane (Δ). The total force acting across the four faces must be equal to zero, since the elastic body is in equilibrium. This total force can be expressed in terms of the values of (T_i) and (T_{ij}) at (P), as well as (l_i) and the area of the sloping face, correct to the lowest order in the dimensions of (T). If (T) is then made to collapse on the point (P), the following relation results:

$$(2.2) \qquad T_i = \sum_{j=1}^{3} T_{ij} l_j.$$

In order to determine the reactions between all pairs of adjacent particles in an elastic body, it is thus necessary only to find the nine stress components (T_{ij}), which are functions of the coordinates.

Let us now consider the box (B) shown in Figure 3. One corner of (B) is at the point (P), and all faces of the box are parallel to the coordinate planes. Since the elastic body is in equilibrium, the forces acting across the faces of (B) must have a zero sum and a zero moment about the origin (O) of the coordinate system. These forces can be expressed in terms of the values of (T_{ij}) at the point (P), correct to the lowest order in the dimensions of (B). If (B) is then made to collapse on the point (P), the following equations result:

$$(2.3) \qquad \sum_{j=1}^{3} \frac{\partial T_{ij}}{\partial x_j} = 0,$$

$$(2.4) \qquad T_{ij} = T_{ji}.$$

The three equations in (2.3) are called the equations of equilibrium. We note that because of (2.4) it is necessary to find only six stress components in order to determine the reactions between all pairs of adjacent particles in the body.

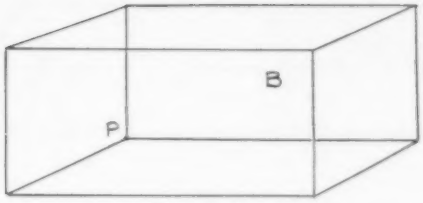

Figure 3

3. A Study of Strain. Let (u_i) denote the three components of the displacement of the general particle (P) in the elastic body. The strain components (e_{ij}) are then defined by the relations

$$(3.1) \qquad e_{ij} = 1/2 \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

The strain components have physical meanings. For example, (e_{11}) is the increase in length per unit length of an element at (P) originally parallel to the (x_1) axis; also (e_{23}) is one half of the decrease in the angle between the elements at (P) originally parallel to the (x_2) and (x_3) axes.

It is interesting to note that $(e_{ij} = 0)$ identically when (u_i) are the components of a rigid-body displacement.

In many problems it is convenient to find (e_{ij}) first, then obtain (u_i) from equations (3.1). These equations (3.1) are then regarded as six equations for the determination of the three unknowns (u_i). These six equations possess solutions only when (e_{ij}) satisfy a set of six second order differential equations called the equations of compatibility. These equations will not be required in our discussions which follow. Accordingly they will not be reproduced here. They are mentioned only for completeness.

4. The Stress-Strain Relations. Since stress and strain go hand-in-hand, there must be some relation between them. The usual form of this relation is given by a law called the Generalized Hooke's Law which is as follows: the stress components are linear functions of the strain components.

We shall assume that the elastic body under consideration is isotropic; this means that all directions in the body are equivalent elastically. In this case the stress-strain relations take the form

$$(4.1) \qquad T_{ij} = \lambda \theta \delta_{ij} + 2\mu e_{ij}.$$

where (λ) and (μ) are elastic constants, and

$$(4.2) \qquad \theta = \sum_{k=1}^{3} e_{kk},$$

$$(4.3) \qquad \delta_{ij} = 1 \quad \text{if } i = j,$$

$$= 0 \quad \text{if } i \neq j.$$

5. The Torsion of a Circular Shaft. Let us consider a shaft with a uniform circular cross-section, the shaft being acted upon by equal and opposite twisting couples (M), as shown in Figure 4. In order to determine the stress and strain in the beam it is necessary to solve equations (2.3), (2.4), (3.1) and the equations of compatibility for the unknowns (T_{ij}), (e_{ij}), (u_i), where the solutions must be such that the proper loading conditions are satisfied over the sides and ends of the shaft.

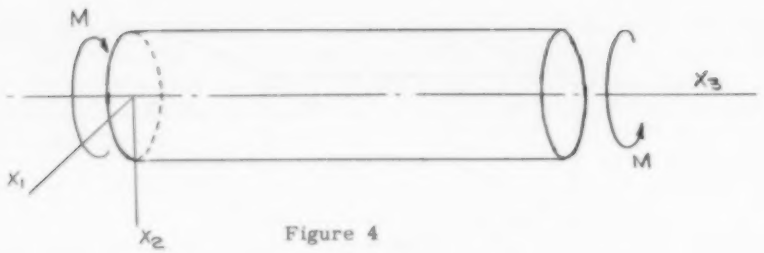

Figure 4

A straight forward attack leads to difficulties, because of the large number of unknowns and the complexities of the equations involved. Accordingly it is customary to make certain simplifying assumptions, then solve the problem, justifying the assumptions by an appeal to the fact it is known that the mathematical problem has but one solution. The simplifying assumptions made in the present case are the following: (a) particles are not displaced out of the cross-section in which they lay initially, (b) each cross-section is rotated as a rigid body through an angle proportional to the distance from one end of the shaft. On the basis of these two assumptions it then readily follows that

$$(5.1) \qquad u_1 = -\alpha x_2 x_3, \qquad u_2 = \alpha x_1 x_3, \qquad u_3 = 0,$$

where (α) is a constant, and is the angle of twist per unit length of the shaft.

Because of (5.1) the strain components (e_{ij}) can be computed from (3.1), and the stress components (T_{ij}) can then be found from (4.1). These stress components satisfy the equations of equilibrium (2.3), as well as the loading conditions over the sides of the shaft. In general they satisfy the loading conditions over the ends of the shaft, only when the twisting couples (M) are distributed over the ends of the shaft in a special way. Hence, when the twisting couples are not applied to the ends of the shaft in this special way, (5.1) is valid everywhere in the shaft except near the ends.

6. The Torsion of a Shaft with a General Cross-Section. Let us now consider the problem of torsion treated in Section 5, but in the case when the cross-section is a general region (S) bounded by a curve (C), as shown in Figure 5. In order to find the stress and strain in the shaft, simplifying assumptions are made just as in Section 5. In the present case these assumptions are: (a) the components of displacement (u_1) and (u_2) are given by (5.1); (b) the axial component of displacement (u_3) is a function of (x_1) and (x_2) only. We can thus write in the present case

$$(6.1) \qquad u_1 = -\alpha x_2 x_3, \qquad u_2 = \alpha x_1 x_3, \qquad u_3 = \alpha \phi \ (x_1, x_2),$$

where (ϕ) is an unknown function of (x_1) and (x_2), called the warping function.

Because of (6.1), the strain components (e_{ij}) can then be computed in terms of (ϕ) from (3.1). From (4.1) the stress components (T_{ij}) can then be found in terms of (ϕ). These stress components satisfy the equations of equilibrium (2.3) if

$$(6.2) \qquad \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} = 0 \qquad \text{in S.}$$

These stress components also satisfy the loading conditions on the sides of the shaft provided

$$(6.3) \qquad \frac{d\phi}{dm} = m_1 x_2 - m_2 x_1 \qquad \text{on C.}$$

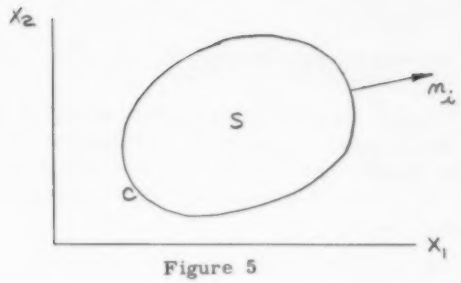

Figure 5

where (m_i) are the components of the unit vector normal to the sides of the shaft, and $(d\phi/dm)$ denotes the directional derivative of (ϕ) in the direction of the vector (m_i). Just as in the case of the circular shaft, these stress components satisfy the loading conditions on the ends of the shaft only when the twisting couples (M) are distributed over the ends of the shaft in a special way. We then have the following conclusion: in order to determine the stress and strain in a beam with a general cross-section and in a state of torsion, it is only necessary to find a function (ϕ) satisfying the differential equation (6.2) throughout the cross-section (S) of the beam, and the boundary condition (6.2) on the boundary (C) of the cross-section. Of course, in general this solution will be valid everywhere except near the ends of the beam.

Any function satisfying the differential equation (6.2) is said to be harmonic. Every harmonic function possesses a conjugate function which is also harmonic and is related to it by certain relations called the cauchy-riemann relations. If (ψ) denotes the conjugate of the warping function (ϕ), these relations are

$$(6.4) \qquad \frac{\partial \phi}{\partial x_1} = \frac{\partial \psi}{\partial x_2}, \qquad \frac{\partial \phi}{\partial x_2} = -\frac{\partial \psi}{\partial x_1}$$

Because of (6.3) and (6.4), we see that the conditions for the determination of (ψ) are now

$$(6.5) \qquad \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} = 0 \qquad \text{in S.}$$

$$(6.6) \qquad \psi = 1/2\,(x_1^2 + x_2^2) + \text{const.} \qquad \text{on C.}$$

The solving of the torsion problem can thus also be reduced to the determination of a function (ψ) satisfying the differential equation (6.5) throughout the cross-section, and boundary condition (6.6) on the boundary of the cross-section. This is so because the warping function (ϕ) can be readily deduced from (6.4), once (ψ) has been determined.

There is a third mathematical formulation of the torsion problem which is of interest. We introduce a function (χ) by the relation

$$(6.7) \qquad \chi = \psi - 1/2\,(x_1^2 + x_2^2).$$

It is then readily seen that the solving of the torsion problem can be reduced to the determination of a function (χ) satisfying the equations

$$(6.8) \qquad \frac{\partial^2 \chi}{\partial x_1^2} + \frac{\partial^2 \chi}{\partial x_2^2} = -2 \qquad \text{in S,}$$

$$(6.9) \qquad \chi = \text{const.} \qquad \text{on C.}$$

HYDRODYNAMICS

7. Irrotational Flow of a Fluid. Let us consider a mass of incompressible fluid in motion. Let us introduce rectangular cartesian coordinates (x_i), as before. Let (P) be a general particle in the fluid, and let (v_i) denote the components of the velocity vector of (P). In general (v_i) are functions of the coordinates of (P) and of time (t). To denote this we write

$$(7.1) \qquad v_i = v_i(x, t).$$

This flow is said to be irrotational if the following relations are satisfied identicall

$$(7.2) \qquad \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} = 0.$$

The study of irrotational flows is usually simpler than the study of rotational flows, since in the former case the determination of the three functions (v_i) can be reduced to the determination of a single function called a velocity potential.

8. Two-Dimensional Flows. A flow is said to be two-dimensional if it is possible to choose coordinate axes so that

$$(8.1) \qquad \frac{\partial v_1}{\partial x_3} = \frac{\partial v_2}{\partial x_3} = v_3 = 0.$$

Let us now consider a two-dimensional flow of an incompressible fluid. Further let the flow be irrotational; this means that

$$(8\ 2) \qquad \frac{\partial v_1}{\partial x_2} = \frac{\partial v_2}{\partial x_1}$$

In addition, let us suppose the flow to be steady; this means that

$$(8.3) \qquad v_1 = v_1(x_1, x_2), \qquad v_2 = v_2(x_1, x_2),$$

i.e., the distribution of velocities throughout the fluid does not change with the time.

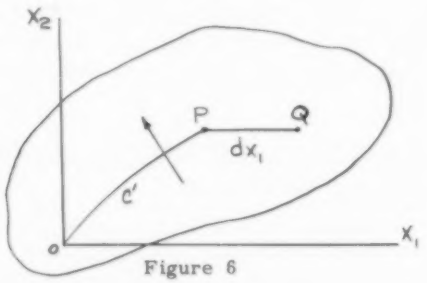

Figure 6

Let (P) be a point in the (x_1x_2) plane, and let (C^1) be any curve joining (P) to the origin (O) as shown in Figure 6. Let us consider the cylinder made up of all the straight lines through (C^1) and parallel to the (x_3) axis. The amount of fluid flowing across a unit length of this cylinder, per unit time, in the direction of the arrow, is denoted by (ψ). Since the fluid is incompressible, (ψ) is a function only of the coordinates (x_1) and (x_2) of (P). It is called the stream function. The lines (ψ) = const. are called the stream lines.

Let us now change the corrdinate (x_1) of (P) by an amount (dx_1), as shown in Figure 6. This produces a change of ($d\psi$) in (ψ), which change is due to the rate of flow across a unit length of the cylinder with (PQ) as its trace. This rate of flow is easily seen to be ($v_2 dx_1$) whence we have

$$v_2 = \frac{\partial \psi}{\partial x_1} \tag{8.4}$$

Similarly, by allowing the coordinate (x_2) of (P) to change by an amount (dx_2) we find that

$$v_1 = - \frac{\partial \psi}{\partial x_2} \tag{8.5}$$

The determination of the two velocity components has thus been reduced to the determination of the single stream function (ψ).

Because of (8.2), (8.4), (8.5) it follows that

$$\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} = 0 \ . \tag{8.6}$$

9. Motion of a Fluid Inside a Rotating Cylinder. Let us now consider the steady irrotational flow of a perfect fluid inside a hollow cylinder which is rotating with a uniform angular velocity (ω) about a line parallel to the generators of the cylinder. Let (S) denote the cross section of this cylinder, and let (C) denote the curve bounding the cross section, as shown in Figure 7.

The motion is characterized by a stream function (ψ) which satisfies the differential equation (8.6) throughout the region (S). It remains only to deduce the boundary condition (ψ) must satisfy on the boundary (C) of the cross section.

Let (P) be a general point on the boundary (C), as shown. The velocity of (P) is (ωr) in the direction shown, and the component of this velocity in the direction normal to (C) is ($\omega r \cos\alpha$), where (α) is the angle between the radius vector to (P) and the tangent to (C) at (P), as shown. Now the velocity in the direction of the normal to (C) of the fluid particle in contact with (P) must also be ($\omega r \cos\alpha$). Hence the rate of flow across a unit length of the cylinder with the element (PQ) of (C) as its trace is ($\omega r \cos\alpha \, ds$), where (ds) is the length of (PQ). But this is equal to ($d\psi$). Thus

$$\frac{\partial \psi}{\partial s} = \omega r \cos\alpha \qquad \text{on C.} \tag{9.1}$$

From Figure 7 it follows that ($\cos\alpha = \frac{dr}{ds}$). Thus

$$\frac{\partial \psi}{\partial s} = \omega r \frac{dr}{ds} \qquad \text{on C,}$$

whence we have the boundary condition

$$\psi = 1/2 \ \omega r^2 + \text{const.} \qquad \text{on C.} \tag{9.2}$$

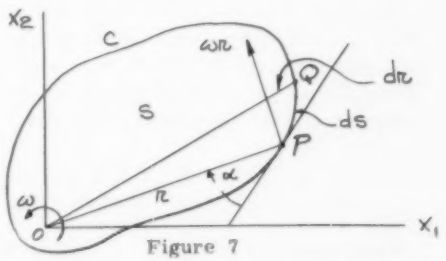

Figure 7

Now $r^2 = 1/2 \ (x_1^2 + x_2^2)$. If we introduce new rectangular cartesian coordinates (y_1), (y_2) defined by the relations

$$y_1 = \sqrt{\omega}\, x_1, \qquad y_2 = \sqrt{\omega}\, x_2 \ .$$

We then see that the problem under consideration is characterized by a function (ψ) which satisfies the differential equation

$$(9.3) \qquad \frac{\partial^2 \psi}{\partial y_1^2} + \frac{\partial^2 \psi}{\partial y_2^2} = 0 \qquad \text{in S,}$$

$$(9.4) \qquad \psi = 1/2\,(y_1^2 + y_2^2) + \text{const. on C.}$$

It should be noted that equations (6.5) and (6.6) for the determination of the torsion function (ψ), and equations (9.3) and (9.4) for the determination of the stream function (ψ), are mathematically identical. This demonstrates one analogy between elasticity and hydrodynamics.

SOAP FILMS

10. The Equilibrium of a Soap Film. Let us consider a thin-walled tube, one end of which is closed by a soap film. Let the region occupied by the soap film be the region (S) shown in Figure 8, the boundary of (S) being a curve (C). Let us now apply a pressure to the air inside the tube, denoting by (p) the excess of the pressure inside the tube over the pressure outside the tube. Of course the soap film bulges outward We shall now consider the equation for the determination of the shape of this soap film.

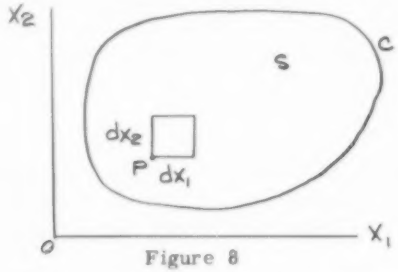

Figure 8

Let (T) denote the surface tension of the soap film. It is the force per unit length acting across any curve drawn on the soap film, and is a constant for any given soap film.

Let (P) be a general point in the undisplaced soap film, and let the coordinates of (P) be (x_1), (x_2). Let (W) be the displacement of (P) normal to the region (S). It is then required to find (W) as a function of (x_1) and (x_2)

Let us consider an element of the undisplaced film at (P), with edges of lengths (dx_1), (dx_2), as shown. This element is displaced into a curved surface in equilibrium under the pressure (p) and surface tensions acting along its four edges. The force acting on the element in the direction of the (x_3) axis and due to the pressure (p) is ($p dx_1 dx_2$). The resultant in this same direction of the surface tension forces acting on the two edges originally parallel to the (x_1) axis can be shown to be

$$T \frac{\partial^2 W}{\partial x_2^2}\, dx_1 dx_2$$

when the assumption is made that the displacement (W) is much smaller than the smallest diameter of (S). Similarly the resultant of the surface tension forces acting on the two edges originally parallel to the (x_2) axis can be shown to be

$$T \frac{\partial^2 W}{\partial x_1^2}\, dx_1 dx_2$$

Hence it follows that the total force acting on this element in the direction of the (x_3) axis vanishes if

$$(10.1) \qquad \frac{\partial^2 W}{\partial x_1^2} + \frac{\partial^2 W}{\partial x_1^2} = - \frac{p}{T}$$

This is the equation of equilibrium of a soap film under small deflection.

On the curve (C), we must have

$$(10.2) \qquad W = O$$

Let us now introduce new rectangular cartesian coordinates $(\bar{z}_1)$, $(\bar{z}_2)$ defined by the relations

$$(10.3) \qquad \bar{z}_1 = \sqrt{\frac{p}{2T}}\ x_1, \qquad \bar{z}_2 = \sqrt{\frac{p}{2T}}\ x_2$$

Then equations (10.1) and (10.2) for the determination of the deflection (W) of the membrane can be written simply as

$$(10.4) \qquad \frac{\partial^2 W}{\partial \bar{z}_1^2} + \frac{\partial^2 W}{\partial \bar{z}_2^2} = - 2 \qquad \text{in S,}$$

$$(10.5) \qquad W = O \qquad \text{on C.}$$

Let us now examine equations (6.8), (6.9) for the determination of the torsion function (χ), and equations (10.4), (10.5) for the determination of the deflection (W) of a soap film. Since the constant in equation (6.9) can be chosen equal to zero with no loss of generality, it thus appears that the problem of determining the torsion function and the deflection (W) are mathematically identical. This demonstrates an analogy between elasticity and soap films.

Because of the above analogy, the following statements can be verified:

(a) The volume swept out by the inflated membrane is proportional to the torsional rigidity of the corresponding shaft in torsion, where the torsional rigidity is the twisting couple required to produce a unit angle of twist per unit length. Because of this property, a simple determination of the comparative torsional rigidities of a number of shafts can be made by experimental work carried out with soap films.

(b) The point on the displaced soap film where the gradient is steepest corresponds to the point in the twisted shaft where the stress is greatest.

Elaborate devices have been constructed for measuring and interpreting the performance of soap films, in order to thereby obtain information about the corresponding problems in elasticity and hydrodynamics.

In conclusion, it should be pointed out that the analogies discussed here are not the only ones existing between these three fields. Further information concerning these analogies can be obtained from the following books: H. Lamb, Hydrodynamics, Cambridge University Press, Cambridge, England, 6th ed., 1932; S. P. Timoshenko, Theory of Elasticity, McGraw-Hill Book Co., New York, 1934.

"On the Importance of Anchorage in the
Applications of Mathematics to Industry."

Richard S Burington[1]

1 Introduction. There is a growing bond between the methods used by men in management and those used by scientists. The potential possibilities of this bond are so great that the power of such developments should be more fully understood. This is a matter of concern for executives, educators, scientists, and engineers. In particular, those engineers, scientists and mathematicians whose interests lie in border fields or in industry should be especially concerned.

Most of us are well acquainted with the contributions scientists, engineers and mathematicians have made in the discovery, research, development, production, and the realization of all sorts of industrial achievements; the improvement of transportation--by land, by sea, and by air--, the wide-spread introduction of labor-saving devices, and the great material progress of recent years.

Many spectacular achievements have been widely publicized; while relatively little is known by the public at large concerning the less spectacular contributions professional men, such as applied mathematicians, have made. These contributions have been wide spread. Each of us is familiar with some specific phases of this progress. Perhaps less is known by the public, and even by the professions themselves, of the contributions professional men--and, in particular, applied and industrial mathematicians--can and have made to the planning, execution, coordination, and administration of large-scale enterprise, and developments and operations of every conceivable type.

These contributions, whether they have involved purely technical research, development, and analysis in the small, or whether they have entailed analysis relating to the planning, execution and administration--that is management--of overall large-scale developments or operations, have in every successful case hinged upon studies which were firmly anchored in the fields involved.

As with the problems of nature, the problems of the industrial mathematician are seldom staged in the manner mathematicians might wish they were. The problems the industrial mathematician must face are commonly those in which a portion of the premises and perhaps a portion of the conclusions are given; and to varying extent portions of the premises and conclusions may be given as data with different degrees of probability, rather than as precisely determined data. To form well-mapped structures joining the assumptions and conclusions, rigorous logical reasoning is needed, but it is the foundation upon which the structure rests, the manner in which it is anchored, which is most important, and which is so often uncertain and hence precarious.

The great importance which must be attached to the proper anchorage of analysis to the underlying fields of investigation cannot be over-emphasized. This is particularly true in the applications of mathematical and statistical methods to the problems of industry, both in the technical as well as in the management fields.

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1. Dr. Burington is Director of the Evaluation and Analyses Group and Chief Mathematician, Bureau of Ordinance, Navy Department. The views expressed in this paper are those of the author and are not necessarily those of the Navy Department. This paper was delivered at a meeting of the Society on May 24, 1949.

2. The formulation of a problem, a fundamental process. Most of you have had the experience of being confronted with studies of an industrial, physical or natural character in which the confusion was almost overwhelming, and have found it extremely difficult to extract from the mass, or dirth, of available information a sensible formulation of the pertinent problems. The larger and more difficult problems of industry are apt to be of this type. Just how to determine what is essential and fundamental is very often distressingly difficult. An overall point of view is usually necessary. Sometimes dealing with the simplest cases helps. But in all cases a thorough understanding of the underlying physical and economic fields is extremely valuable. Frequently, many attempts are necessary. Occasionally, unconventional treatments are fruitful. The product of such considerations is a certain simplification and idealization of the original situation and formulation of the problem at hand. It is such idealized models which are made the subject of analysis.

The use of approximate equivalent systems appears as part of a fundamental process in almost all studies of physical, economic and industrial phenomena. One method of approach consists broadly of the following steps:

(1) The extraction from the phenomena or situation S of a nearly equivalent physical (economic, or other) model P.

(2) Reduction of the physical (or other) model P to an approximate equivalent mathematical model M of amenable treatment. This may involve the selection of a set of observable quantities which are assumed to be appropriate for the studies at hand.

(3) A study of mathematical model M involving various relationships R between the observable and perhaps non-observable quantities, with solutions of pertinent items relating to the system M.

(4) The interpretations of the solutions found in (3) in terms of the mathematical model M.

(5) The interpretation of the solutions found in (4) in terms of the physical (or other) model P.

(6) The interpretation of the results (5) in the original physical (or other) settings S.

Steps (1) or (2) constitute the formulation of the foundation for the mathematical studies involved. If these foundations are not firmly anchored in the underlying basic setting S, then the results obtained in (5) and (6) may be properly questioned.

Steps (3) and (4) are properly those for the mathematician as such. It is his business to reveal all that is implied by the relationships R. This step may not always be possible with the present state of the art. However, the mathematician may then be forced to make further modifications of the model M with the attended dangers to the soundness of his formulation and anchorage in the original phenomena S. Such situations sometimes lead to important researches in pure mathematics.

It is of considerable importance to realize that idealized mathematical model M is somewhat different from the actual prototype situation S. For this reason mathematical results derived from the model M and subsequently interpreted in S may and should be questioned as to their validity, applicability and relevance to the original setting S. Thus the anchorage of M in S must be the subject of thorough investigation, in the light of its bearing on the relevance and applicability to the original and

primary setting S of the mathematical results derived from the model. All of us who deal with applied and in particular indusfrial and business problems, know that if the model differs too greatly from the original situation S, any related practical considerations and projects may be delayed or brought to naught. No matter how logical or beautiful an analysis is made in connection with an industrial consideration, if it fails to meet the realistic tests of relevance, adequacy and applicability, sooner or later it will be recognized by those engaged in the original larger problems as being inadequate and impractical and may even be called worthless.

To determine whether a model falls within allowable bounds must, as a general rule, be determined by experiment, after the analysis or at least a portion of it has been considered. One crucial test as to the worth of a theory or design, deduced mathematically or otherwise, hinges on the degree of agreement or disagreement at suitable points with experiments and measurements. This usually entails a consideration of the reliability of the test data and the reliability of the premises upon which the analyses were made. Thus, assuming the absence of actual logical or calculating errors, and the legitimacy of the pure mathematical processes used, the inferences to be drawn from the final judgment of the worth of a theory or design bear upon the legitimacy of the abstractions and simplifications which were made in constructing the model and anchoring it in the original setting S.

Another crucial test of a theory, design, or particular model structure lies in the power of the theory to predict observations in the prototype field previously unknown, and in the ability of the theory to include all known pertinent facts in the universe it is designed to describe.

If, in addition, a particular theory or model structure proves to be suggestive and stimulating to new developments, it is very likely to turn out to be a truly successful theory, and perhaps more so than some other apparently suitable working theories which are less suggestive.

In examining the typical equations developed or encountered in mathematical physics--and in industry--one soon realizes that they relate partly to physical objects and events, and partly to a mathematical structure which was introduced for reference purposes. Further study of such equations will reveal a distribution between those quantities called observables, which can be determined by a specified observational procedure, and those quantities which are not observables. The non-observables may depend at least partly on a mathematical framework which cannot be the subject of actual observation. The non-observables may be very useful for computational and theoretical or descriptive purposes. However, such non-observables must be removed in the final stage where actual predictions must be checked by observational verifiable results. A consciousness of this aspect of anchorage is most important.

The importance of experiment as an aid in the formulation of a well-anchored mathematical framework for a field of study should not be overlooked. Consider for example the old problem of attempting to understand and predict cavitation, say in the neighborhood of a propeller, turbine blade, or other body immersed in a fluid. Many attempts to understand the growth and collapse of cavitation bubbles have been made because of their bearing on the performance of various machines, etc. Were one to make a new start in formulating such a theory of cavitation, one should study every bit of experimental data available: for example, the reports and photographs made at a very high speed of the formation and collapse of bubbles from tests recently made in tunnels at the California Institute of Technology. From this one should try to draw a certain mechanistic picture of the phenomena which could be colored with known thermodynamic and hydrodynamic knowledge. In this way the construction of a well-anchored structure for an improved theory of cavitation would begin.

It is very important to experimental science that theories be proposed. Without theory much experimentation would be fruitless, or poorly directed. As a tentative theory is formulated, particular attention to its agreement with observations should be made. A slight inconsistency of theory with observations may lead to new discoveries which (without a theory) might otherwise remain unnoticed. These discrepancies may lead to an improved theory, and so on; but always the anchorage of theory to observations must be kept in mind.

The industrial mathematician, scientist and engineer in the course of his work sometimes becomes concerned with a mathematical structure of a theory for an applied field, which is considered in the light of its predictions as a basis for making important designs, and which may involve considerable financial, personnel or material risk. In such cases the experienced qualitative or quantitative judgments of others should be sought, a judgment which will take into consideration factors quite apart from the particular mathematical structures used in the work, a judgment which may take account of the reliability of data, the relative importance of various factors and other pertinent items beyond the scope of the mathematical structure available, each with appropriate weights. In making final decisions--as must be done in industrial problems--a careful synthesis of all of the approaches and methods available should be made to ensure the best possible decision.

3. Examples illustrating the importance of strong anchorage. I have just sketched an outline of a fundamental process familiar to all of us, the process of formulating a working theory or model for the purpose of studying some field, physical or otherwise; and in industry, often for the purpose of developing, producing or improving a product, a process, an operation, or the like. This art of formulating a problem in a specific field of application is usually the more difficult, and often it is more important than its actual solution.

In order to illustrate this process and to discuss in more detail the importance of good anchorage in the study of industrial of other problems, I shall discuss briefly certain examples which are suggestive of the importance of the theme of this lecture.

Non-linear analysis. There are many cases of importance to industry where the physical and mathematical formulations for a situation are fairly well established, but the purely mathematical theories and techniques needed do not exist and consequently should be developed. The field of non-linear analysis furnishes many current examples, stemming from problems of applied mathematics in the fields of dynamics, hydrodynamics, aerodynamics, etc. I refer to problems in which the usual principles of superposition do not hold; problems in which the equations encountered are not linear and which when approximated by linear means lead to solutions which are incompatible with the underlying physical problems. The development of mathematical methods and techniques to cope with such problems is the subject of much research these days. Improved methods for producing reliable predictions will be developed in time. In the meanwhile the industrial and applied mathematician must do the best he can to cope with such problems and to construct the simplified models in such a way that he can make the most reliable predictions possible.

As a familiar example of a type of problem in this field, I mention a class of problems relating to the motion of ships. In such problems one form of a differential equation which must be studied in order to understand the behavior of ships is

$$\ddot{x} + (a\dot{x} + b\dot{x}^2) + cx = 0 , \tag{1}$$

where x is the angle of roll of a ship in still water. When this equation is approxi-

mated by an equation of the form

$$\ddot{x} + a\dot{x} + cx = 0 , \tag{2}$$

and solutions of (2) are obtained for certain boundary conditions, it is found that the solutions so predicted are completely at variance with the known behavior of a ship of the type and conditions under study. The replacement of equation (1) by equation (2) for purposes of calculation and study oftentimes disrupts the soundness of the anchorage so that the solutions deduced are practically worthless in the field of study.

As another example, I cite the familiar Van der Pol equation

$$\ddot{x} + k(x^2 - 1)\dot{x} + x = 0 , \tag{3}$$

which is encountered in the study of the simple vacuum tube. Theory shows that on the slightest provocation the current x must go into a well-defined oscillation. Explicit solutions of this equation (3) are often too complicated to be useful or else are not known. For these reasons the approximation

$$\ddot{x} - k\dot{x} + x = 0 , \tag{4}$$

is sometimes used. However a study of the solutions of equation (4) indicates that the current x will increase indefinitely once it has been started. This, of course, is contrary to the facts. Again the trouble lies in an inadequate anchorage of the mathematical model, equation (4), in the actual field under study.

Use of simulators. In the development of such devices as automatic pilots and control systems; in the study of the electrical and mechanical stability of interconnected systems; and in the study of aerodynamic stability, the typical dynamical situations involved often lead to non-linear differential equations of difficult types. Attempts to replace these equations by linear systems commonly lead to grossly incorrect predictions as to the behavior of the physical systems under consideration. In order to avoid the pitfalls of such poor anchorage, engineers, scientists and mathematicians have resorted to the construction of various types of simulators to carry out the desired solutions. Such methods have been fruitful, but they have also frequently failed because of the failure of one or more of the components of the simulator to faithfully simulate the desired equations. This is again a failure due to poor anchorage of the system in the original setting of the problem. Experts in this field all agree that when recourse is made to the use of simulators to solve such problems, special care must be made to insure suitable anchorage; and such experts go farther by recommending that an independent dynamic analysis be made of the problems which the simulators are designed to solve in order to furnish as far as possible a sound foundation for and check of the mathematical and physical systems involved.

Bridge design. The engineering mathematicians concerned with the design of structures and bridges have over the generations found much to interest them. To design a safe, economical, durable, and useful bridge is an art few men master. The history of bridge design and construction reveals many tragic instances where the lessons learned in one generation have been forgotten in another. In every major case of failure in modern times, it appears that the mathematical foundation upon which the design was generated was faultily anchored in the physics of the problem. The Takoma Bridge disaster is a recent example. Here it appears that the underlying equations did not take adequate account of the stability of the bridge under aerodynamic forces in relation to its flexibility and other characteristics.

In applying the various theorems and formulas of a branch of mathematics or statistics to an applied field, it is very essential to make sure that all of the mathematical premises underlying the formulas of theorems are properly satisfied in the field of application. Violations of this principle can easily lead to serious errors in the primary field of application. It was the violation of this principle which led the older classical dynamics and hydrodynamics into a veil of disrepute. The underlying mathematical formulas used sometimes did not take adequate account of such things as viscosity, compressibility and the like; and as a consequence predictions made using such formulas and related mathematical theorems were sometimes seriously in error.

Serious errors can be made in using the formulas of statistics by not paying strict attention to the theoretical premises underlying their validity and use. It is very often a temptation to use some statistical formula, say, for confidence limits, without being sure the theoretical premises underlying the use of such limits are properly met in the underlying field. The improper use of such formulas may result in a completely false picture of the reliability of the data and of the associated predictions for the field of study.

The applied mathematician is commonly confronted with a system whose state S_1 is expressed by functions of observables which are known with varying degrees of probability. It is sometimes necessary to predict the effect of various influences on the system and its state. The solution is likely not to result in a unique state, but rather in the form of an expected state. In other words the solution may be given in the form of a state S_p which may be expected to occur with a probability p.

When dealing with such situations the applied mathematician may be inclined to feel he is working with less rigorous mathematics. But, such a point of view can be costly, for he must be very careful that the probability formations he uses are based on as reliable a theory and data as can be obtained.

4. Nature of pure mathematics in applications. When pure mathematics is considered as an axiomatic deductive system, the development of the theory is considered to proceed from a set P of definitions and postulates--the selection being largely a matter of choice--which are not subject to proof within the theory. P is stated in terms of certain primitive concepts left undefined in the theory. The entire theory is completely determined when the premises and primitives have been stipulated and the principles of logic to be used have been selected. Every theorem of the theory can then be derived logically from P. Pure mathematical propositions are of the form P implies T where P and T involve parameters x, y, $\cdots$. The mathematics is not concerned with the truth of P or T, but rather it is concerned with assertions of the type "if P is true, then T is true."

This point of view with regard to pure mathematics leads one to raise the question as to why mathematics has proved itself of such value in the applied sciences and in industry. In any field of application the role of mathematics is to render explicit propositions Q and assumptions in the field of study which are implied by the premises P underlying the treatment of the problem, the propositions Q being hidden and perhaps heretofore unnoticed. Thus arises the source of the loose statement: "the power of mathematics to discover new facts." Actually the results obtained are only those implied by the premises.

When the terms of a pure mathematics are given physical definitions, and the postulates in their physical interpretations can be considered true, then the theorems of the mathematics in the light of their physical interpretation can be taken as necessarily true. Such theorems may tell us that if D_1, D_2, $\cdots$ are done, then S_1, S_2, $\cdots$ will happen.

The actual function of mathematics in any industrial or applied field is clarifying, analytic, but not strictly within itself predictive. The mathematics furnishes a technique. The predictive power originates in the initial fundamental laws or premises assumed for the field under study. If one makes full and proper use of mathematical theory and techniques, the failure of a prediction in a field S must be attributed to the unsuitability and inadequacy of the premises and observational data involved--that is to poor anchorage of the mathematical structure in the field S.

5. The use of the concept of equivalence. The concept of equivalence lies at the heart of nearly every physical and mathematical system. In the study of physical systems, perfect or approximate isomorphic systems play a fundamental role. Equivalent physical and mathematical models are integral parts of the structures used.

One abstract formulation of the concept of equivalence is as follows:

Let A, B, C, $\cdots$ be elements in the fixed system S being considered (such as abstract entities, quantities, $\cdots$), and suppose a possible relationship that may occur between these elements A, B, C, $\cdots$ is defined in some manner or other. Suppose that when A and B are so related, the relationship is indicated by the symbol ξ, and the expression $A \xi B$ is written; and that when A and B are not so related, the fact is indicated by the symbol $A \not\xi B$. Then the relationship expressed by the symbol $A \xi B$ is known as the equivalence relation provided that it is defined to satisfy the following four properties:

I. Determination. For any pair of elements A and B of S the relation $A \xi B$ either holds or does not hold.

II. Reflexivity. For any A one has $A \xi A$.

III. Symmetry. When $A \xi B$, then $B \xi A$.

IV. Transitivity. When $A \xi B$ and $B \xi C$, then $A \xi C$.

A is said to be equivalent to B if $A \xi B$ is an equivalence relation.

Every such definition of equivalence, which is ordinarily not unique for the given system, constitutes a division of the elements A, B, C, $\cdots$ into classes.

All of us are familiar with mathematical examples illustrating equivalence. Thus, in euclidian geometry two polygons A and B may be equivalent in the sense that they have equal areas, but they may or may not be equivalent in the sense that they have the same number of sides or angles. Again, any two proper conics A and B are equivalent in the sense that one may be transformed into the other by a projective transformation; while on the other hand, A may or may not be equivalent to B in the sense that A may be transformed into B by means of a rotation and a translation.

In general, in geometry or physics, whether two quantities are equivalent may be determined by calculating certain numbers called invariants which are associated with the object under study.

A related mathematical concept of considerable importance in its own right, as well as in its application, is that known as isomorphic systems.

Consider two systems A and B each consisting of a set of elements and a set of operations on these elements. Suppose that each system is closed with respect to a given system of operations in the system. The two systems A and B are said to be

isomorphic, or abstractly identical, with respect to these operations if there exists a bi-unique one-to-one correspondence between the elements of A and B, such that any formal combination of, or operation on, the elements in A corresponds to the analogous construction with the corresponding elements in B.

In any specific study of physical phenomena, as in the applications of mathematics, the elements a, b, c, $\cdots$ of a system S are identified with specific physical objects, numbers, quantities, entities, or the like, and the definitions of equivalence used must be carefully given in terms of these elements and the physical systems to which they belong. Each definition of equivalence used constitutes a separation of the set of elements a, b, c, $\cdots$ and associated physical phenomena or systems into classes. This separation into classes may be of considerable physical significance, each type of equivalence often having associated with it an extensive physical theory.

Both in mathematical theories and in physical theories many non-isomorphic types of equality have been and can be defined. Each type of equality may have associated with it an extensive theory. Thus, in the theory of equivalent linear electrical circuits, two 2-pole networks may be equivalent in the sense that for all frequencies they have identical driving-point admittances (or more generally, for 2N-pole networks, that they have identical characteristic coefficient admittance matrices), yet they may not be equivalent in the sense that they have the same number of independent mesh circuits; or, are structurally the same; or, are equally economical to operate; or, are both readily physically realizable; and that if A and B, respectively, are their network matrices, A and B may or may not be equivalent in the sense of matric congruence, $\cdots$.

In the design of structures, ships, dams, and many other things, it is necessary to make accurate predictions as to the characteristics, cost and performance of various proposed designs. Often it is necessary to use models in this type of work. The results of the studies with the models are then used to predict the performance and characteristics of the proposed prototype. Great care must be taken in using such methods, and the mathematical structures involved must be studied carefully with respect to their anchorage, both in the model and in the prototype.

Theoretically, in designing a model test and in constructing the models for the test, every physical parameter of importance in the prototype must be considered and taken into account. It is not sufficient merely to make the dimensions of the model and prototype proportional. If a theory of similitude is not available, then one must be developed. Such a theory must serve as the basis for designing and constructing the model from its prototype dimensions. It must serve also as the guiding principle in interpreting the measurements made on the model in terms of the corresponding measurements as predicted for the prototype. If this theory of similitude is a perfect one, and the application of this theory is also perfect, then there is a one-to-one bi-unique correspondence between the various physical parameters of the prototype and the corresponding ones of the model; and every operation on or with the prototype has a corresponding operation on the model. In other words, any characteristic of the model as predicted through the isomorphism thus stated must be a true characteristic of the prototype, and vice versa. (This is an example of the equivalence property, if $A \xi B$, then $B \xi A$.)

In practice it is impossible to apply perfectly such a theory and expect the results predicted through the model test to be completely true for the prototype. A measure of this perfection lies in a comparison of the actual characteristics of the prototype with those predicted by the model studies. Consequently the aim in all such work is the construction of as nearly perfect an equivalent system of models as is humanly possible. In other words, these models must be as nearly isomorphic with their corresponding prototypes as possible.

In one type of theory of similitude which is widely used in aero- and hydrodynamics a certain set of dimensionless numbers

$$\pi_1, \pi_2, \cdots, \pi_n \cdots$$

are used which must be kept unchanged if true similarity is to exist between the characteristics of the prototype and its model. In practice the dimensionless numbers π_i of concern are generated from quantities measuring important physical quantities involved in the study. For example, where inertia and gravity forces play an important role, the Froude number has been commonly used as one of the dimensionless numbers. If true similarity is to exist, every dimensionless parameter π_i referring to conditions in the model must have the same numerical value as the corresponding parameter for the prototype. This commonly means that the model and prototype must be completely similar geometrically. From a practical standpoint it is usually impossible to preserve the invariance of all of the dimensionless numbers π_i for the model and prototype. For this reason the experimenters, as well as the designers and mathematicians concerning themselves with such problems, must pay unusual attention to the nature of the similarities or dissimilarities, and must watch carefully the foundation of their mathematical structure in the physics of both the model and the prototype.

Thus, the use of equivalent systems appears as a fundamental process in almost all studies of physical phenomena and in the applications of mathematics to various phenomena. Because of the hazards involved in the use of such systems, the industrial mathematician must necessarily pay great attention to the tie-in of his mathematical structure with the underlying phenomena under study.

6. Evaluation and analysis of systems. In recent years we have witnessed the emergence of a new field of applied mathematics, various branches of which are known as operations research, evaluation and analysis of systems, and the like. This field is capable of very wide industrial application. All branches of the armed services have made wide use of such fields as a very necessary and useful adjunct to the management, planning, and development of large-scale operations and projects and as an aid in making important decisions involving large expenditures of money, material and human effort. It is in these fields that I have devoted much of my time in recent years. The foundations of these fields are well established in some respects but not in others.

It is only natural that such methods are being increasingly used by management since the acts and decisions of executives in a closely-knit industrial organization such as we have in this country tend to have very deep and acute influences upon the course of events, the flow of production, as well as upon industry as a whole and the individual. The resources of this country have reached such a state that few industries can any longer afford to use intuition, hunches, and the like, when scientific and analysis methods are available which will reduce the risks involved. The margin of profit within which industry now must operate is so slim that it is imperative that careful analyses be made relating to all important steps.

Most large scale industry has much need for evaluation groups which have as their function:

(1) Clarification of objectives.

(2) Clarification of an operational or production process or status.

(3) Analysis of plans, projects, operations, existing methods, and development

programs with comparative studies and predictions which indicate the directions in which improvements might be made with existing facilities and modes of operation.

(4) Analyses which indicate the need for exploring new fields and new methods.

(5) Analyses and predictions which indicate the directions in which investigations or projects should be pointed in order to realize the optimum desired results.

(6) Analyses and predictions which exhibit the courses of action open, together with studies of what such actions entail.

(7) Aid in the formulation of policies.

(8) Aid in the technical direction of various tasks, projects and the like.

In this kind of work which bears upon management decisions, it is extremely important that the analyses and evaluations be well anchored in the particular fields to which they apply. Where poor anchorage is involved, the price may be extremely great to the industry, which usually cannot afford costly mistakes. Consequently, extreme care must be made in formulating the foundations for such studies. This requires a deep knowledge of the fields involved and the best logical approaches that can be devised. The use of incomplete or questionable measures of merit, the oversimplification or the over-generalization of a study are apt to be dangerous in this field of endeavor.

In spite of the difficulties, no large-scale industry can afford to do without adequate scientific evaluations and analyses. In this field applied mathematicians, as well as engineers and scientists as a whole, can contribute much, particularly when they concentrate diligently on the anchorage of their work in the field of study.

7. Conclusions. To summarize, I have discussed in a broad way some of the problems which the applied mathematician must face in industry, and have placed particular emphasis on the great importance which he must attach to the anchorage of his work in the underlying fields I have outlined a certain fundamental process common to most applications of mathematical methods. In this process the use of equivalent systems appears as a fundamental process. Unless the foundations of such systems are firmly rooted in the primary field of study, the results are most apt to fail--without proper anchorage the work of the applied mathematician will fail and perhaps lead to costly errors.

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THE LOGIC OF PROGRAMMING ELECTRONIC DIGITAL COMPUTERS

by Arthur W. Burks[1]

1. The Machine Language.

The preparation of a problem for an electronic digital computer involves two steps: the statement of the problem in a form suitable for digital computation (e.g., the replacement of differential equations by difference equations)[2] and the formulation of instructions to the machine (coding or programming). It is the aim of the present paper to explain the fundamental principles governing this second step.

We are concerned here with large-scale, general-purpose electronic digital computers. Such computers consist of four parts: the memory (which stores both the numerical data of the problem and the instructions to solve it), the logical-arithmetical unit (which performs arithmetical and logical operations on the data and instructions), the input-output unit (which communicates information to and from the machine and serves as an auxiliary memory), and the control (which decodes or interprets the instructions and causes the computer to execute them).[3]

An electronic computer solves a problem by performing specified operations (e.g., integration, interpolation, sorting) on given data, numerical or alphabetical (e.g., initial conditions for difference equations, function tables, ledger accounts). For the expression of this information in the computer a machine language is required.[4]

We will assume a language in which the basic symbols are decimal digits; these are organized into aggregates of fourteen, each called a numerical word, or, more briefly, a number. Alphabetical information can obviously be expressed in numerical words.

Now a computer will do only what it is instructed to do, so in order to state a problem to it we must provide it (in addition to the data itself) instructions for manipulating the data. For this purpose command words, or commands, must be included in the machine language. (Numbers other than data are also used for instructing the machine, as will be clear later.) Numbers and commands are called indifferently words. Though every word is a number, in a sense, some are interpreted as numbers proper and some as commands or instructions.

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[1] A paper delivered to the Industrial Mathematics Society, Detroit, Michigan on May 25, 1950. Arthur W. Burks is Professor of Philosophy, University of Mich.

[2] A discussion of topics connected with this step may be found in Douglas R. Hartree, Calculating Instruments and Machines, University of Illinois Press, Urbana, Illinois, 1949. Ch. 9.

[3] For a more complete description of a typical computer see A. W. Burks, H. H. Goldstine, and John von Neumann, Preliminary Discussion of the Logical Design of an Electronic Computing Instrument, Institute for Advanced Study, Princeton, New Jersey, 1947.

[4] We are not concerned here with the design of machine languages, but only with how to use them. Hence we will describe the machine language to be employed in this paper without discussing its merits. It should be emphasized that this language was chosen merely because it seems to the writer to be convenient for illustrating the ideas of this paper.

The commands are made to refer to the words to be operated on in the following way. Addresses are assigned to the areas (bins) of the memory which store words. Let us assume that the memory can hold 1,000 fourteen-digit words; the bins will then be assigned the addresses 0 through 999. If we allot two digits of a command word to indicate the operation (addition, division, etc.) then there is room in the remaining twelve digits to specify four of these three-digit addresses.

We can now describe the commands.[1] There is one for addition which is written

$$+ (x, y, z, w) \quad .$$

It instructs the control to add the number at address x to the number in bin y (without destroying either of these numbers) and place the answer at address z (erasing what is already there), then to go to address w for the next command to be executed. Thus the first three arguments of the command have to do with the arithmetical operation itself and the last argument with executing a sequence of operations of which it is a part. A good notation for expressing the arithmetical portion of the addition command is

$$[x] + [y] \rightarrow [z]$$

where "[x]" means the contents of bin x, and the arrow signifies that the sum of the contents of bins x and y is to replace the prior contents of bin z.[2] For example, if bins 1, 2, and 3 held the numbers 42, 97, and 323 respectively before the command "+ (1, 2, 3,)" was executed, they would hold the numbers 42, 97, and 139 respectively after the command was executed.

By means of the above notation we can easily describe the remaining arithmetical commands:

$- (x, y, z, w)$	$[x] - [y] \rightarrow [z]$
$X (x, y, z, w)$	$[x] \cdot [y] \rightarrow [z]$
$\div (x, y, z, w)$	$[x] \div [y] \rightarrow [z]$
$\sqrt{\ } (x, z, w)$	$[x] \xrightarrow{1/2} [z]$
$T (x, z, w)$	$[x] \rightarrow [z]$

As in the addition command, the last argument in each command ("w") specifies the location of the command to be executed next. We will assume that the control will start a problem (when so instructed by the operator) by executing the command at address 0. Thereafter each command it executes will tell it where to find the

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[1] We give here only the internal commands, omitting those for ordering information into and out of the machine. The reason for this is that, in the interest of brevity, we wish to confine our attention to what goes on inside the machine. We shall also ignore such factors as decimal points and ranges of numbers. In the actual use of a machine, of course, all of these things are very important and add to the complexity of the programming. The general principles of programming, however, stand out more clearly in the less complex situation we deal with here.

[2] This is a modification of a notation proposed by Hartree, Op. cit., p. 113.

next command. The computation may be stopped by using the command

F

which instructs the control to stop executing commands and to signal the operator (e.g., by a light) that the problem has been solved.

Though these commands cover the elementary arithmetical operations to be performed, two other commands, called logical commands, are needed in programming. The conditional switch of control command

C (x, y, z, w)

instructs the control to find its next command at z or w accordingly as $[x] < [y]$ or $[x] \geq [y]$. The substitution command

S (x, y, i, z)

orders the control to modify the command at y by substituting in its i'th argument place (i = 1, 2, 3, or 4) the number at address x, and then to go to z for the next command. These are called logical commands because they involve the logical operations of conditional switching and substituting needed to control and organize the computation.

2. The Logic of Programming.

The operations which an electronic computer is built to perform are in general much more complicated than those expressed by the nine commands just listed. Some typical operations are: integrating, differentiating, interpolating, computing logarithms, looking up values in a function table, sorting, posting to a ledger, and arranging data monotonically. Since the machine does not understand directly commands for such operations as these, it is necessary to instruct it in detail as to how to perform them. This is done by providing it with sequences of commands and numbers, called program sequences or programs. The activity of constructing such sequences is called programming. It is the major purpose of this paper to develop principles of programming, that is, to show how programs may be constructed out of commands, numbers, and other programs.[1]

Our task can be simplified by taking seriously the fact that a machine language is a language. It is, of course, an artificial language, expressly designed for a particular computer, in contrast to a language like English, which is a natural language that has developed gradually without much explicit construction. There are, nevertheless, important similarities in structure between the machine language and the English language. Thus consider how we make commands in English: e.g., "Add the first twenty numbers on the board." Such a command is compounded out of smaller linguistic parts - the words. Now in a broad sense a program in the machine language is a command or imperative to the machine to perform some more or less complicated operation, and it also is compounded out of smaller linguistic parts (numerical and command words). Thus instructing a machine is like commanding a human being in that we use a language with a limited vocabulary of words and express different commands by combinations of these.

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[1] For this reason our examples have been chosen for their didactic value rather than for their realism.

Now a problem to be programmed is generally first formulated in ordinary English and mathematical symbolism. In the next section, for example, we are given the task of programming the imperative "Add all the numbers from a_1 to a_I." It follows that programming is a kind of translating: the function of the programmer is to translate a set of instructions or commands from ordinary language into the language the machine understands. And from this it in turn follows that we will be aided in our analysis of programming by any available relevant knowledge concerning the structure of languages. Modern symbolic logic provides us with some information of this kind, which we shall utilize.

That mathematical logic is relevant to programming will no doubt appear surprising to many readers, as perhaps does the use of "logic" in the title of this paper. Admittedly, the subject matter of traditional logic (rules for deducing true conclusions from true premises) is not relevant here, since commands are neither true nor false and are not deduced from one another as are declarative sentences. However, contemporary logicians investigating the nature of valid inference have found it necessary to study the grammar or syntax of language - i.e., how compound symbols are constructed out of their linguistic parts (primitive symbols). As we have just seen, a program is a combination of words (in the machine language) which constitutes a compound imperative. Hence our problem is in important respects similar to one of the problems the mathematical logician studies. It is therefore natural that some of the concepts employed in logic (e.g., propositional function, free variable, bound variable, and quantifier) are useful in the theory of programming.

We must emphasize that there is an important difference between machine languages and the artificial languages usually investigated in symbolic logic. The latter are declarative languages,[1] whereas machine languages contain imperatives (commands); indeed, it is with commands that programming is mainly concerned. However, as we have already indicated, we are not presenting a logic of imperatives in the sense of providing rules of deduction for commands. Rather -- taking advantage of the similarity of the grammar of imperatives to the grammar of declaratives -- we are applying logical concepts developed for declaratives in our analysis of imperatives.

3. The Structure of a Simple Program.

In this section we will analyze a simple program, and in the subsequent sections show how such simple programs may be compounded to give more complex ones. Suppose we wish the machine to perform the simple summation

$$(1) \qquad \sum_{i=1}^{I} a_i$$

What we want the machine to do is expressed in ordinary language by the command "Add together the numbers a_1 through a_I." Since, however, there is no (single) command in the machine language for repeated summation, we must make the imperative more explicit before we can translate it into this language.

One way of forming the required sum is to begin with the quantity 0 and successively add the a's to it. Such a process is called a recursion or induction and is best

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[1] Various logics of imperatives have been proposed, but the writer did not find them of any aid in the problem at hand.

formulated by means of a recursively defined function. Define the function g_i recursively by

$$g_0 = 0$$

$$g_i = g_{i-1} + a_i$$

Then

$$g_i = \sum_{j=1}^{i} a_j$$

and g_I is the desired sum. The imperative to be communicated to the machine is thus

(2) For every integer i, in succession and starting with i = 1 and

$g_0 = 0$,

if $i \leq I$ then compute g_i by adding g_{i-1} to a_i.

Our task, then, is to translate this statement into the machine language. We shall do so by analyzing it into three parts and translating one part at a time. The three parts are:

(a) For every integer i, in succession and starting with i = 1 and $g_0 = 0$

(b) if $i \leq I$ then...

(c) compute g_i by adding g_{i-1} to a_i.

We shall program each of these in turn, but before doing this let us compare them with corresponding parts found in a declarative sentence of similar structure. Consider

Every man is mortal,

which is more explicitly expressed as

(2') For every x, if x is a man then x is mortal.

This contains the linguistic parts

(a') For every x

(b') if ... then ...

(c') x is mortal

which are closely similar to those of the imperative (2). We shall take advantage of these similarities in our programming.

Let us program (c) first. Consider (c'). In symbolic logic "x is mortal" is called a propositional function because it determines a functional correspondence between names (e.g., "Socrates," "this chair") and definite propositions (correspondingly, "Socrates is mortal" and "This chair is mortal"). Similarly we can distinguish between command functions (e.g., "add g_{i-1} to a_i") and definite commands (e.g., "add 6 to 15"). ("Command" will be used indifferently to refer to either.) Of course, the control understands and will execute only definite commands, not command

functions. For example, "+ (18, 6, 3, 139)" can be expressed in the machine language as a fourteen-digit number, but "+ (x, y, z, w)" cannot since the machine language described in Section 1 contains no variable symbols.[1] We did employ variables to explain the machine language, since we wished to describe the action of the computer for any definite command of a given form. For the same reason command functions and ordinary numerical functions as well (e.g., "g_i" in bin 1 below) are essential in programming. Hence we will extend the concept of machine language to include such expressions. This will cause no confusion as long as it is remembered that before a program is actually given to the machine (i.e., before the task of translation is fully completed) the variable symbols should be removed (see Program II infra).[2]

To program (c), then, we must write a command function in the machine language which is a translation of the English command function "compute g_i by adding g_{i-1} to a_i." Suppose that the a_i's are stored in the bins L + 1, L + 2, ..., L + I (so that a_i is stored at L + *i*) and g_i is stored in bin 1. Then the following is a translation of (c) into the machine language:

1. g_i

2. + (1, L + i, 1, _) $\qquad\qquad g_i = g_{i-1} + a_i$

We may call this part of the program the computation part. The symbols on the left describe (by means of the variables) the contents of the bins at each stage of the computation. Note that the variable "i" is chosen so as to describe the contents of the bins at the end of the execution of this part of the program rather than at the beginning. The notation on the right is to help the programmer keep account of the role the various words play in the computation.

Because the control understands and executes only definite commands, not command functions, we must instruct it to specify properly all command functions into definite commands before it reaches them. This, in fact, is what is commanded by (a). (a) tells the control to keep account of i, the number of times the command function at address 2 is executed, and for each value of i to make the proper substitution into this command function. A translation of (a) into the machine language is thus:[3]

3.	i	
4.	1	Constant
5.	L	L + i is the location of a_i
6.	_	Temporary storage
7.	+(3, 4, 3, 8)	$i \longrightarrow i-1$.
8.	+(3, 5, 6, 9)	6: L + i
9.	S(6, 2, 2, 2)	

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[1] A language containing variable symbols and which the machine can in a sense understand is suggested in the last paragraph of Section 5.

[2] The situation may be described more precisely in this way: the programmer uses a metalanguage containing variables to talk about an object language (the language understood by the machine) which does not contain variables. It is unnecessary for us to distinguish these two languages in our subsequent exposition, however.

Since the function of this part of the program is to specify properly the command functions in the computation part into definite commands, we will call it the specification part of the program. The last command is written so that the control will switch to the computation part of the program when it has completed this part.

We have already noted the similarity of (a) to (a'). In logic "for every x" is called a quantifier since it states the quantity of cases - in this instance all - for which the propositional function following it ("if x is a man then x is mortal") holds true. Similarly, (a) instructs the machine to execute the subsequent command function (c) an indefinite number of times. It is the purpose of (b) to stop the computation at the proper place. (b) consists of the conditional connective "if ... then ..." together with the criterion for continuing the computation. A translation of (b) into the machine language thus involves the conditional shift of control command and I:

10. I

11. C(3, 10, 7, α) If i < I then control to 7
If i = I then control to α.

α is left unspecified since we do not know yet where the control should go after this computation is finished. We will call this part of the program the conditional part since its function is to control the repetition of the cycle in accordance with the given conditions.

To obtain a single program which is a translation of (2) into the machine language we must complete the interconnections of the three parts by specifying the last argument of the command at 2. Since the conditional part of the program should be executed after the computation part, command 2 should read:

2. +(1, L + i, 1, 11) .

The sequence of words in the bins 1 through 11 and L + 1 through L + I constitutes the program for the problem specified. We shall number it Program I.

Actually Program I does not constitute a definite set of instructions to the machine. This is the case because such quantities as g_0, L, and the a's have been left unspecified. In the terminology used earlier Program I is a command function, not a definite command. (Cf. the original imperative (2) in this respect.) Since it seems best to reserve the word "command" for the simple commands introduced in Section 1 we will say that Program I is a program function and any complete specification of it is a definite program. ("Program" is used indifferently to refer to either.) To instruct a machine to solve a problem we must, of course, give it a definite program, since only definite programs (not program functions) are understood by the control.

We offer as an example of a definite program the following specification of Program I, which we shall call Program II:

1. 0 Initial value of g_i

2. +(1, _, 1, 11) $g_i = g_{i-1} + a_i$

---

3 This sequence of words does not quite constitute a complete translation of (a) since it does not fix the initial values of i and g. This is done by specifying the contents of bins 1 and 3 to be 0 before the computation commanded by (2) starts.

3.	0	Initial value of i
4.	1	Constant
5.	11	L
6.	_	Temporary storage, initial contents irrelevant
7.	+(3, 4, 3, 8)	i ⟶ i-1
8.	+(3, 5, 6, 9)	6: L + i
9.	S(6, 2, 2, 2)	
10.	4	I
11.	C(3, 10, 7, 16)	
12.	123	a_1
13.	247	a_2
14.	181	a_3
15.	49	a_4

If we add to this sequence of words commands for getting to it from it--

0.	C(7, 7, 7, 7)	
16.	F	

--and express all of the words as fourteen-digit numbers in the machine language, we will have a complete problem ready to give to the machine. The machine will start with the command at 0 and then switch to the command at 7.[1] It will cycle the commands 7, 8, 9, 2, 11 in that order four times and then go to 16 and stop, signalling the operator that the problem is finished. At this time address 1 will hold 600, the answer to the problem. The reader may find it helpful actually to follow the control around the feedback loop a few times, noting how the contents of bins 1, 2, and 3 vary during the computation.

It is important to notice what happened to command 2 in the specification of Program I into a definite program. A blank is written in the second argument place since no relevant number could be placed there: no matter what number is placed there the program will substitute the correct value of "L + i" in its stead at the proper time. Thus in converting Program I into a definite program we did not assign, and could not significantly assign, values to these occurrences of the variables "L" and "i". Since the specification part of the program does this, Program II is a definite program despite our failure to specify these variables. (Cf. in this regard the fact that "For every x, if x is a man then x is mortal" is a definite proposition even though it contains three occurrences of the variable "x".) In contrast, the occurrence of the variable "L" at 5 must be specified to convert Program I into a

1. Note that the conditional switch of control command becomes an unconditional switch of control whenever all the arguments are identified. There are other ways of producing the same result, e.g., in this case "C(3, 3, 8, 7)" and "C(5, 4, 7, 7)."

definite program. Employing the terminology of symbolic logic we can say that the occurrences of the variables "L" and "i" at 2 are bound occurrences in Program I whereas the occurrences of the variables "L" and "I" at 5 and 10 are free occurrences in Program I.[1] (Cf. the ordinary mathematical distinction between variables and parameters.)

Note that whether an occurrence of a variable is bound or free is relative to a specified sequence of words. Thus though the occurrence of "i" at 2 is bound relative to Program I, it is free relative to the computational part of this program; indeed, it is the function of the specification part of the program to specify it.[2] Since such occurrences of variables are specified by parts of a given program function we need not specify them to convert that program into a definite program. Rather, we have only to specify all free occurrences of variables in that program function. The following occurrences of variables are free in Program I and hence were assigned values to convert this program function into definite Program II: "L" in 5, "I" in 10, "α" in 11, "a_i" through "a_I" and the initial values of "g_i" and "i" in 1 and 3 respectively.[3]

The problem we have programmed is, of course, too simple a problem for a general purpose computing machine to bother with. It illustrates many principles of coding, however, since a typical definite program will be made up by compounding a number of such program functions. In constructing programs and combining them to make more complicated programs it is convenient to have diagrams which show the structure of each program and the path of the control through its parts. Such diagrams are called flow diagrams.[4] The flow diagram for Program I is shown in Figure 1. It consists of three boxes, corresponding to the three parts of the program (called the specification box, the computation box, and the conditional box, reading from left to right) interconnected by directional lines showing the path of

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1 The application of this logical disinction to programming was first made by H.H. Goldstine and John von Newmann in Planning and Coding of Problems for an Electronic Computing Instrument, Institute for Advanced Study, Princeton, New Jersey, Vol. I, 1947, p.10. See also Vol. II, 1948 and Vol. III, 1948. The writer is heavily indebted to this work in regard to the general theory of programming.

2 In terms of the original comparison between (2) and (2') we can say that the translation of (a) binds some of the variables in the translation of (c) very much as the quantifier (a') binds the variable in (c').

3 We have treated these initial values as free variables even though they were bound (specified) in (2). It is convenient to handle the initial values this way since they must be specified anyhow every time a program function is used. (A typical program function will be used several times during a problem.)

4 This conception and terminology are also owing to von Neumann and Goldstine, Ibid., Vol. I p.4.

5 It should be noted that we coded the computation box before the specification box, though the control passes through them in the reverse order. This procedure is especially convenient if many free variables occur in the computation part of the program, for then the specification of these can proceed systematically and all together. It might seem desirable on this account to reverse the order of the computation and specification boxes, but to do so would be to unbind (make free) many of the variables in the computation part of the program and thus increase the labor of specifying the program function into a definite program.

the control through the program from the entry point 7 to the exit point α. A double arrow is used in the specification box to symbolize the fact that it is the function of this part of the program not only to advance i by 1 for each cycle but also to specify all occurrences of free variables in the computation part of the program that must be respecified for each new value of i (in this case "i" and "L" in 2). (See footnote [5] on page 44.)

4. Integrating a Trajectory.

Let us now program a more complicated problem to show how the principles established in Section 3 may be used. The problem the first general purpose electronic computer (the ENIAC) was built primarily to solve was that of the trajectory, and it will serve as a good example.

We begin with the differential equations for the path of a projectile in motion:[1]

$$x'' = -Ex'$$

$$y'' = -Ey' - g$$

where

$$E = \frac{e^{-hy}\, G(s)}{C} \quad , \quad s = \sqrt{(x')^2 + (y')^2},$$

x and y are the horizontal and vertical positions, a prime denotes differentiation with respect to the independent variable (t), g and h are fixed constants, C is a constant for a given shell, and G(s), the ballistic drag function, expresses the resistance of the air to the shell as a function of its speed. These may be reduced to the following set of simultaneous first order equations:

$$u = x'$$

$$v = y'$$

$$u' = -Eu$$

$$v' = -Ev - g$$

$$E = \frac{e^{-hy}\, G(\sqrt{u^2 + v^2})}{C}$$

To get an approximate solution to these equations by digital computation we replace them by corresponding difference equations. These are, symbolizing e^{-hy} by "F(hy)" and the time interval of integration by "Δ":

$$s_{n-1} = (u_{n-1}^2 + v_{n-1}^2)^{1/2}$$

$$E_{n-1} = \frac{F(hy_{n-1})\, G(s_{n-1})}{C}$$

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[1] For the derivation of these equations see G. A. Bliss, Mathematics for Exterior Ballistics, John Wiley and Sons, New York, 1944, Ch. 2.

(3)

$$t_n = t_{n-1} + \Delta$$

$$x_n = x_{n-1} + u_{n-1}\Delta$$

$$y_n = y_{n-1} + v_{n-1}\Delta$$

$$u_n = u_{n-1} - E_{n-1}\, u_{n-1}\, \Delta$$

$$v_n = v_{n-1} - E_{n-1}\, v_{n-1}\Delta - g\Delta$$

It will be noted that these equations are recursively formulated, so that given the values of the appropriate quantities at the beginning of an interval (i.e., t_{n-1}, x_{n-1}, y_{n-1}, u_{n-1}, and v_{n-1}) we can by a series of arithmetical operations produce the corresponding quantities for the end of the interval (i.e., t_n, x_n, y_n, u_n, and v_n).

There are two functions involved in the calculation. The drag function G(s) is determined empirically and hence is provided in the form of a function table. The exponential function F(hy) is normally produced by solving a differential equation as the computation proceeds, but in order to illustrate how the same program function can be used more than once in a single cycle of computation we shall assume that it too is given in the form of a function table. We then need a program to look up values of any given function f(z) in a table, so let us write that next.

We will assume that values of f(z) are given for successive integral values of z beginning with 0 and are stored in sequence beginning at address M. If we assume further that z is an integer[1] then f(z) is located at M + z. The following program function (Program III) will look up the value of f(z) in the given table:

21.	z	Argument
22.	M	Location of f(0)
23.	f(z)	Function value to be found
24.	+(21, 22, 21, 25)	21: M + z
25.	S(21, 26, 1, 26)	
26.	T(M+z, 23, β)	Transfer f(z) to 23 and shift control to β.

It is represented by the flow diagram of Figure 2, which has a specification box and computation box in that order.

We can now program the trajectory. As the equations show, the first step is to determine $F(hy_{n-1})$ and $G(s_{n-1})$. Hence we must command the machine first to specify Program III into a definite program to look up $F(hy_{n-1})$. The control should then

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[1] We make these assumptions to simplify the program even though they are not realistic. For the same reason we do not interpolate and we use only first order difference equations (3) instead of ones of a higher order. In actual practice efficiency with regard to computation time and memory space would require us to do all these things. But including them would add to the complexity of the program without introducing any new principles of coding, and as we pointed out before it is more important for our examples to illustrate basic principles clearly than to be realistic.

switch to Program III and after finding $F(hy_{n-1})$ it should switch back to the main routine to specify Program III into a definite program a second time, this time to look up $G(s_{n-1})$. After passing through Program III again the machine is finally ready to compute t_n, x_n, y_n, u_n, and v_n. This whole process should be repeated until some criterion is satisfied. Suppose we wish to continue the integration until the shell "hits the ground," i.e., until y becomes negative (or zero). Then a conditional command to control the cycling will conclude the program. The structure of the whole program (Program IV) is represented by the flow diagram of Figure 3. Note that, as this diagram makes clear, Program III is included in Program IV as a part. β_1 and β_2, the two points to which the control returns after executing Program III, will be determined as the program is written.

By using the flow diagram we can write the program in a straightforward manner. Let us assume that 100 values of F(z) are stored at addresses 100 through 199. The occurrences of free variables in Program III to be specified to make this into a definite program are "z" at 21, "M" at 22, and "β" at 26. ("M" and "z" in 26 are bound and hence need not be specified.)

31.	h	
32.	y_n	This bin will hold y_{n-1} at the beginning of this step of integration, and y_n at the end.
33.	100	Value of M for F(z)
34.	β_1	Location of command to which control returns after looking up $F(hy_{n-1})$
35.	X(31, 32, 21, 36)	$hy_{n-1} \longrightarrow$ z in 21
36.	T(33, 22, 37)	$100 \longrightarrow$ M in 22
37.	S(34, 26, 3, 24)	$\beta_1 \longrightarrow \beta$ in 26

The last command switches the control to Program III. Since the first command is located at 35, $\gamma = 35$.

This same process is to be repeated for $G(s_{n-1})$, after $F(hy_{n-1})$ has been transferred to temporary storage. Let us assume that 100 values of G(z) are stored in bins 200 through 299.

38.	u_n	This will be u_{n-1} at the beginning of the step of integration.
39.	v_n	This will be v_{n-1} at the beginning of the step of integration.
40.	_	Temporary storage
41.	_	Temporary storage
42.	200	Value of M for G(z)
43.	β_2	Location of command to which control returns after finding $G(s_{n-1})$
44.	X(38, 38, 40, 45)	40: u_{n-1}^2

45. X(39, 39, 41, 46)	41: v_{n-1}^2
46. +(40, 41, 41, 47)	41: s_{n-1}^2
47. $\sqrt{}$(41, 21, 48)	$s_{n-1} \longrightarrow z$ in 21
48. T(23, 40, 49)	40: $F(hy_{n-1})$
49. T(42, 22, 50)	$200 \longrightarrow M$ in 22
50. S(43, 26, 3, 24)	$\beta_2 \longrightarrow \beta$ in 26

Note that $\beta_1 = 44$.

The execution of the second specification of Program III will leave $G(s_{n-1})$ in 23. We can now proceed to compute t_n, x_n, y_n, u_n, and v_n, using the difference equations already given:

51. t_n	This will be t_{n-1} at the beginning of the cycle.
52. x_n	This will be x_{n-1} at the beginning of the cycle.
53. Δ	
54. C	
55. g	
56. $u_{n-1}\Delta$	
57. $v_{n-1}\Delta$	
58. +(51, 53, 51, 59)	$t_n \longrightarrow t_{n-1}$
59. X(38, 53, 56, 60)	56: $u_{n-1}\Delta$
60. X(39, 53, 57, 61)	57: $v_{n-1}\Delta$
61. +(52, 56, 52, 62)	$x_n \longrightarrow x_{n-1}$
62. +(32, 57, 32, 63)	$y_n \longrightarrow y_n^{-1}$
63. X(40, 23, 40, 64)	40: $F(hy_{n-1})$ $G(s_{n-1})$
64. $\div$ (40, 54, 40, 65)	40: E_{n-1}
65. X(40, 56, 41, 66)	41: E_{n-1} $u_{n-1}\Delta$
66. _(38, 41, 38, 67)	$u_n \longrightarrow u_{n-1}$
67. X(40, 57, 41, 68)	41: E_{n-1} v_{n-1} Δ
68. X(55, 53, 40, 69)	40: $g\Delta$
69. +(40, 41, 41, 70)	41: E_{n-1} v_{n-1} $\Delta+g\Delta$
70. _(39, 41, 39, 72)	$v_n \longrightarrow v_{n-1}$

This completes the part of the programming represented by the computation box of Figure 3. Note that $\beta_2 = 58$.

Finally, we code the conditional box:

71. 0	Constant
72. C(71, 32, 35, δ)	If $0 < y_n$ then control to 35 If $0 \geq y_n$ then control to δ .

This part of the program illustrates an interesting point. The path of the control is determined here by one of the answers to the problem (y_n) rather than by a number introduced primarily for purposes of control (e.g., the control number i in Program I). Thus there is a feedback from the numbers being processed to the commands. A like feedback occurs in Program III: what definite command the command function at 26 becomes depends on the data. In general, the path the control follows through the memory depends on the numbers being processed as well as the commands and control numbers which instruct and guide the computation. It is for this reason that we defined "program" to include words expressing data and answers as well as words (commands and control numbers) whose function is to direct the computation. A program is thus a sequence of words which contains instructions for its own transformation.

Program IV does not constitute a complete formulation of the exterior ballistics problem. Apart from the fact that we oversimplified the process of determining the functions G(s) and F(hy) and used only a first order integration technique, Program IV is only a program function, not a definite program. In actual practise it would be employed repeatedly for different initial conditions (muzzle velocities, angles of fire, etc.) and different shells. Hence it would need to be embedded in a program of wider scope which would bind the free occurrences of its variables: "u_0," "v_0," "Δ," "δ," etc. This program would also need to provide for reading in data and writing out the answers, as well as for transforming the trajectories computed into firing tables, the end result of the computation. Though it is beyond the limits of this paper to code the complete program, we shall make a few remarks about the construction of such programs.

5. Program Composition.

Our discussion of how best to construct complicated programs will be facilitated if we can refer to a further example. Suppose we wish to instruct the machine to compute

$$(4) \qquad \prod_{I=1}^{J} \left(\sum_{i=1}^{I} a_{i,I} \right)$$

We can most simply code this problem by using Program I to compute the quantity in parantheses and adding to it a program corresponding to the multiplication symbol "π". As before, the problem is best formulated in inductive terms. Define the function h_I recursively by

$$h_0 = 1$$

$$h_I = h_{I-1} \cdot \left(\sum_{i=1}^{I} a_{i,I} \right)$$

Then

$$h_I = \prod_{k=1}^{I} \left(\sum_{i=1}^{k} a_{i,k} \right)$$

and h_j is the desired sum.

This example is interesting because it involves two quantifiers and hence its computation involves a double recursion. Since, in the formula to be computed, the multiplication symbol "π" is of wider scope than the summation symbol "Σ" we need a program with a feedback loop which will enclose the feedback loop of Program I. The flow diagram of the required program is shown in Figure 4. Program I can be made a part of this program by specifying α in the command at 11 to be the location of the first command (in order of execution) of the computation box of Figure 4. The specification box of Figure 4 is the only one that needs special comment.

In general, the function of a specification box is to bind the free occurrences of certain variables in a particular sequence of words. (Similarly, the function of a quantifier in logic is to bind the free occurrences of a variable in a particular sequence of symbols.) In this case these are the free occurrences of the variable "I" and of the other variables ("L" and the initial values of "i" and "g_i") which need to be respecified for each new value of "I" in the program between the point marked "7" and the end of the conditional box in Figure 4. Thus the specification part of the program must keep a count of the major cycling (i.e., of I) and for each major cycle specify the occurrences of "L" (note that the location number L is now a function of I) in 5 and the initial values of "i" and "g_i" in bins 3 and 1. If any of these variables were free in the computation and conditional boxes of Figure 4 they would have to be specified there also, as would be the case if, for example, the comparison made by the conditional box involved a quantity whose location was a function of I.

This last example and Program IV illustrate a general method of constructing programs: that of coding program functions and later combining these to give larger programs. We can take the fundamental element in this construction to be the pattern shown in Figure 4. Other patterns may be regarded as degenerate cases of patterns like this (e.g., Figure 2 lacks the conditional box and the feedback loop) or as compounds of them in which the elements are hooked together at the exit and entry points (e.g., in Figure 4 the "7" in the exit circle following the specification box and the "α" in the entry circle preceding the computation box connect the diagram of Figure 1 into this diagram.)

Let us consider in more detail this method of programming. Most problems solved on a computer can be broken down into a number of sub-problems. For example, the computation of firing tables involves interpolating, solving difference equations, using a function table, printing, etc. Program functions can be written for these fundamental processes and then used repeatedly. Since these programs will be located in different parts of the internal memory for different problems, most of the addresses must be written as free variables. When a specific problem is to be solved it is not programmed as if it were unique but is dealt with in the following way. First, the program functions corresponding to the fundamental processes involved in solving the problem are selected from a library of programs. Second, a statement is made in the machine language of the unique features of the problem; e.g., a description of a particular set of equations is given and the initial conditions

listed. Finally, a combining program[1] which will bind all of the free variables (including the addresses) of the sub-programs is added to these programs to give the desired definite program.

There is a very practical reason for employing this method of instructing a computer. An electronic digital computer is extremely fast; for example, the ENIAC can compute a trajectory faster than the shell itself can fly. It is important that the increased speed of computation provided by electronic machines be matched by a corresponding increase in the speed of programming. This is possible if programs are constructed by combining already prepared program functions. It should be noted that the specification of program functions into definite programs is not accomplished by rewriting the programs by hand outside of the machine, but is done by the machine under the direction of the combining program. This is significant, for since most of the addresses in the programs are represented by free variables the task of specification is a substantial one. Thus a large part of the work of program composition is done by the machine.

We shall conclude this paper by briefly suggesting a way of using the machine to an even greater extent in program composition. What we have said about programming can be summarized as follows. The programmer begins with a formulation of his problem in the ordinary language of mathematics and English (e.g., (1), (2), (3), (4)). He then breaks down the given statement into elementary parts and translates each of these parts into the machine language; for example, Program I is a translation of a part of expression (4). Finally, he writes a combining program to direct the machine to combine the resultant program functions in a way determined by the arrangement of the parts of the original statement of the problem; for example, since the "π" of (4) is outside of (1) the programmer makes Figure 4 enclose Figure 1.

Under this procedure each program synthesis is treated by the programmer as a unique problem. Now it seems possible to instruct the machine by means of a single translation program to synthesize many definite programs under the direction of the original statements of the problems. Suppose we constructed a programmer's language in which expressions like (1) and (4) could be written. Such a language would be much closer in structure and symbolism to the ordinary language of mathematics and English than the machine language described in Section 1, and hence easier for the programmer to use in stating problems to the machine. The machine could be taught to receive instructions in this language in the following way. The symbols of this language could be represented in the machine language. The programmer would translate simple expressions in this language (such as (1)) into program functions in the machine language. Finally, he would prepare a translation program (in the machine language) to instruct the machine to combine these program functions under the direction of expressions like (4) to make definite programs. Thus we can teach the machine to make the translation from the programmer's language to the machine language, and once this is done the more convenient programmer's language can be used for instructing the machine to solve problems.

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[1] See Goldstine and von Neumann, Ibid., Vol. III.

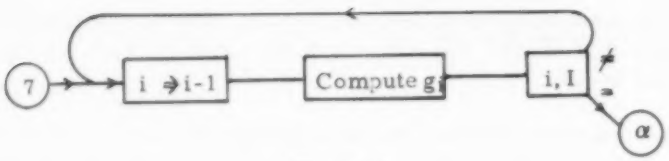


Figure 1

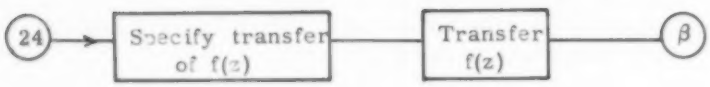


Figure 2

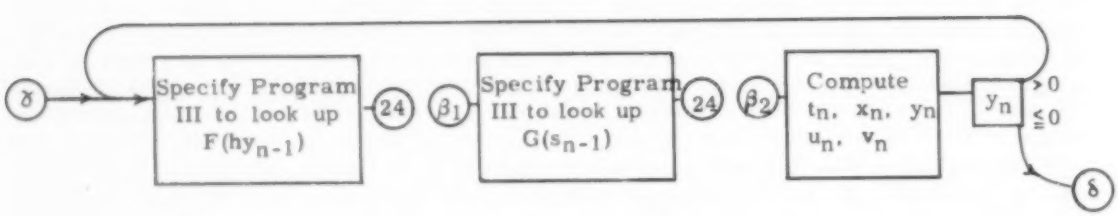


Figure 3

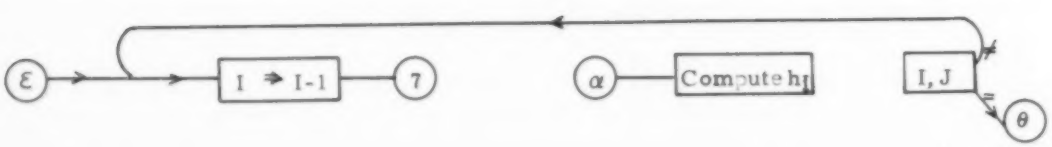


Figure 4

ANALOGUE COMPUTATION

Arvid W. Jacobson [1]

The subject for the February (1950) meeting of the Society was Analogue Computation and its Engineering Applications. The meeting was followed by four conference sessions at which Professor S. H. Caldwell, Director of the Center of Analysis, M.I.T., led the discussions. The meeting also served as an occasion for the opening of the Wayne University Computation Laboratory. (Wayne University had earlier received as gifts from M.I.T. the original differential analyzer built by Dr. Vanevar Bush and the cinema integraph.) A total of 500 local engineers and research people attended this series of meetings.

The conferences began with the description of the differential analyzer, and continued with the discussion of its application to the solution of problems in mechanical vibrations, in electrical transients and servomechanisms and concluded with a session on scientific problems. In this report of the conferences it is not feasible to reproduce the discussions in full. However, we shall briefly describe the differential analyzer, show how problems are set up on it and solve one or two engineering problems. (See bibliography references at the end of the paper for more detailed discussion.)

DESCRIPTION OF THE DIFFERENTIAL ANALYZER

The differential analyzer is a machine for solving ordinary differential equations. In it the value of a given variable is represented by the amount of rotation of a shaft. It thus operates on the physical quantity of angular rotation, and hence is called an analogue machine. The shafts are so interconnected that they rotate in accordance with the functional relations expressed in the differential equation.

The operation involved in the solution of differential equations is integration. The basic component of the machine is a unit which mechanically performs this operation.

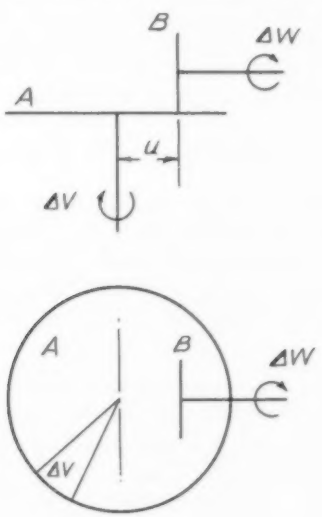


Figure 1

[1] Dr. Arvid W. Jacobson is Assistant Professor of Mathematics at Wayne University and was President of the Industrial Mathematics Society for the season 1949-1950.

A glass disc A is mounted on a vertical axis and a sharp-edged steel wheel B rests on the disc. See Figure 1. If the disc is rotated through angle Δv, the rotation of the wheel is proportional to the product $u\Delta v$, that is

$$\Delta w = ku\Delta v. \quad \text{Hence } w = k\int_{v_o}^{v} u\, dv.$$

The disc is mounted on a carriage displaced by the input motion "u" and is rotated by the input "v". The two inputs "u" and "v" produce an output "w" which is a constant times the integral of "u" with respect to "v". This is shown schematically in Figure 2.

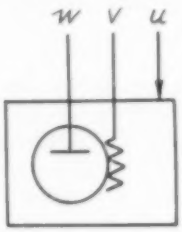


Figure 2

As the friction between the wheel and the disc provides a very small torque to shaft "w", it is necessary to amplify it. This is accomplished by means of a mechanical torque amplifier which consists of friction bands wrapped around rotating drums driven by an electric motor.

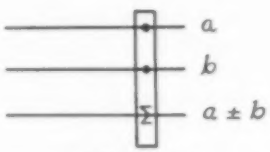


Figure 3

The operation of algebraic addition is performed by means of differential or adder gears (Figure 3). Constant coefficients are introduced by gear ratios and scale factors. While the machine is equipped with a multiplying unit, multiplication can be performed with the integrators, for if P and Q are variables then

$$PQ = \int PdQ + \int QdP.$$

Since the differential analyzer obtains solutions of differential equations, any function of a single variable which is a solution of a differential equation can be generated. For example, solving the equation $d^2y/dx^2 + y = 0$, yields the function sin x. Any function which can be plotted can be fed into the machine from one of the four input tables (Figure 4). While the machine moves a carriage, equipped with an index,

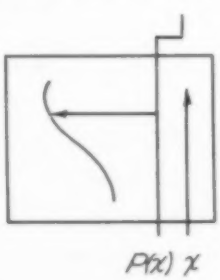
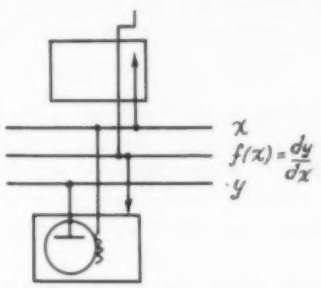


Figure 4

across the table, a crank is operated manually to keep the index on the curve. Thus the value of the function P(x), represented by the shaft labelled P(x), is continuously fed into the machine by the operator. The fact that discontinuous and non-linear functions or any experimental data which can be plotted can be introduced into the machine enables it to solve problems for which the classical method would be impractical.

HOW PROBLEMS ARE SET UP ON THE MACHINE

Let us set up the problem $dy/dx = f(x)$, $y(0) = a$ on the machine. The independent variable shaft x ("x" represents the amount of rotation of this shaft) is driven by an electric motor. The shaft f(x) is turned manually at the input table. Hence we have two inputs "x" and f(x) into the integrator and the output "y" is then the $\int f(x)dx$. The initial condition $y(0) = a$ is introduced as an initial setting of the integrator.



(Dots denote cross gear boxes)

Figure 5

The solution represented by the rotation of shaft "y" may be taken to an output table by means of cross shafts and plotted, and it may also be connected to a counter and printed.

Another example.

$$\frac{d^2y}{dx^2} = h(x) + g(y)$$

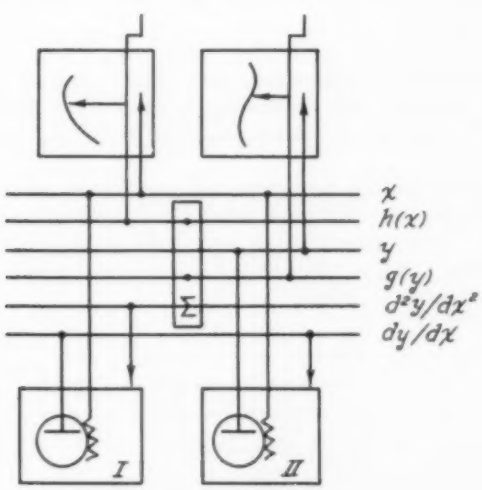


Figure 6

A motor turns the independent variable shaft "x". Since "y" is the solution of the equation, and hence initially unknown, we anticipate the solution and therefore the motion "y". Thus we obtain the motions f(x) and g(y) from the input tables, which in turn drive the adder whose output is d^2y/dx^2. The output of integrator I is dy/dx, and that of integrator II, is "y". Hence the motion which was anticipated is actually provided by integrator II. This is a mechanical feedback and the second integrator "closes the loop". This feedback is also a machine equivalence of equality of the two members of the equation and constrains the variable quantities involved to assume values in accordance with the functional relations expressed by the equation.

In a machine layout of a problem it is essential that one and only one source of torque is applied to each shaft. In the above example the shaft "x" is driven by a motor. The torque to shaft h(x) is provided by an operator, similarly for shaft g(y). The adder combines these two torques into one and supplies it to the shaft d^2y/dx^2. Next, the output of integrator I provides (through a motor driven torque amplifier as explained earlier) the torque to shaft dy/dx. Finally, integrator II supplies the anticipated torque to shaft "y".

In a machine solution of a problem it is necessary to keep the displacements of integrators and function units within design limits. Likewise good precision requires that the number of revolutions of each shaft be fairly large, while the solution time should be as low as possible. These conditions are satisfied by proper determination of scale factors, which we shall illustrate by a simple example.

The notation Ax on a shaft means that the shaft makes A revolutions per unit of "x". Hence if A is 256 and "x" varies by 10 units, the shaft will make 2560 revolutions.

Let us calculate the scale factors in generating e^x by solving the differential equation dy/dx = y. Let the scale factor of the independent variable be A, and that of the function e^x, B. The gear ratio between the two e^x shafts is "n". Then

$$(1)\quad Be^x = \frac{1}{32} \int (Bne^x)\, d(Ax),$$

where the fraction 1/32 is the integrator constant "k". If "x" varies by 5 units, and if the solution time is set at 10 minutes while the maximum speed of the independent variable motor is about 600 rpm, we get an approximation for A: 5A/600 = 10. We choose A = 1024. From equation (1) we get $Be^x = ABne^x/32$, so that An = 32, or 1024 n = 32 and n = 1/32.

The limits of the integrator displacement are ± 40 turns of the leadscrew. Hence the integrand Bne^x must satisfy the following inequality: $Bne^x \leq 40$. With the above values for "n" and "x", we get a maximum value for B, i.e., $B \leq 40 \times 32/e^5$.

The solution on the differential analyzer is generated in a closed loop. Therefore, the error in the solution is also generated in a closed loop and has the same form as the solution itself. The machine is equipped with units to correct for backlash, known as frontlash units. The differential analyzer, under proper conditions, yields results accurate to one part in 1000.

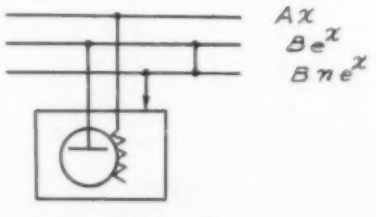


Figure 7

ELECTRICAL CIRCUIT PROBLEM

The circuit is shown in Figure 8, where R is a non-linear function of the current "i". The equation for this circuit is

$$L\frac{di}{dt} + Ri + \frac{1}{c}\int i\,dt = E$$

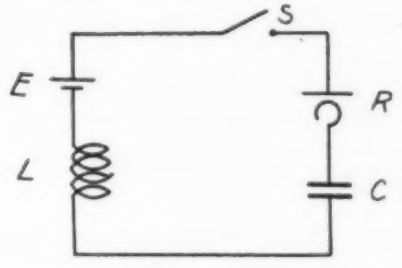
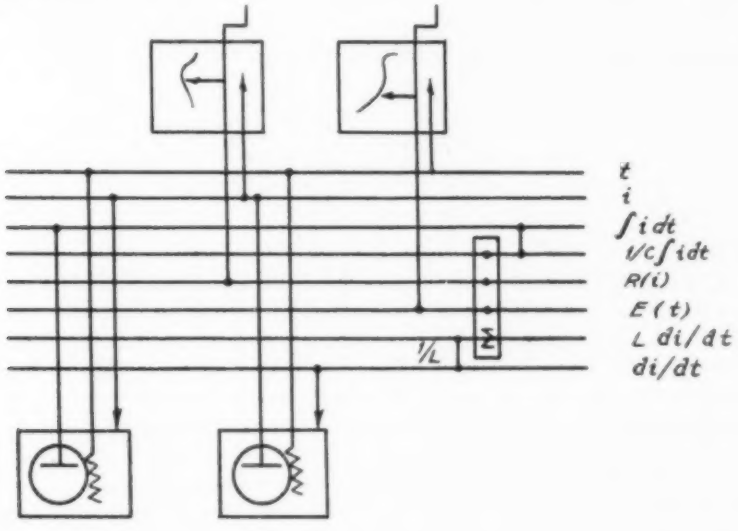


Figure 8

If S is closed at t = 0, we are to find "i" as a function of time. The machine assembly is given below.



SUSPENSION PROBLEM

In order to keep the discussion from becoming too involved we shall consider a vibrating system with only two masses. The arrangement of masses, springs and damping units of the physical problem is shown in Figure 9, where $x_0(t)$ represents the exciting force. The equations of motion are

$$M_1\ddot{x}_1 + c_1(\dot{x}_1 - \dot{x}_0) + c_2(\dot{x}_1 - \dot{x}_2) + k_1(x_1 - x_0) + k_2(x_1 - x_2) = 0$$

$$M_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = 0$$

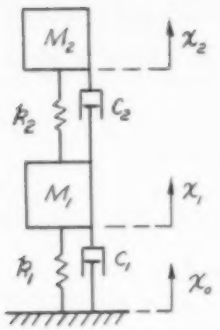


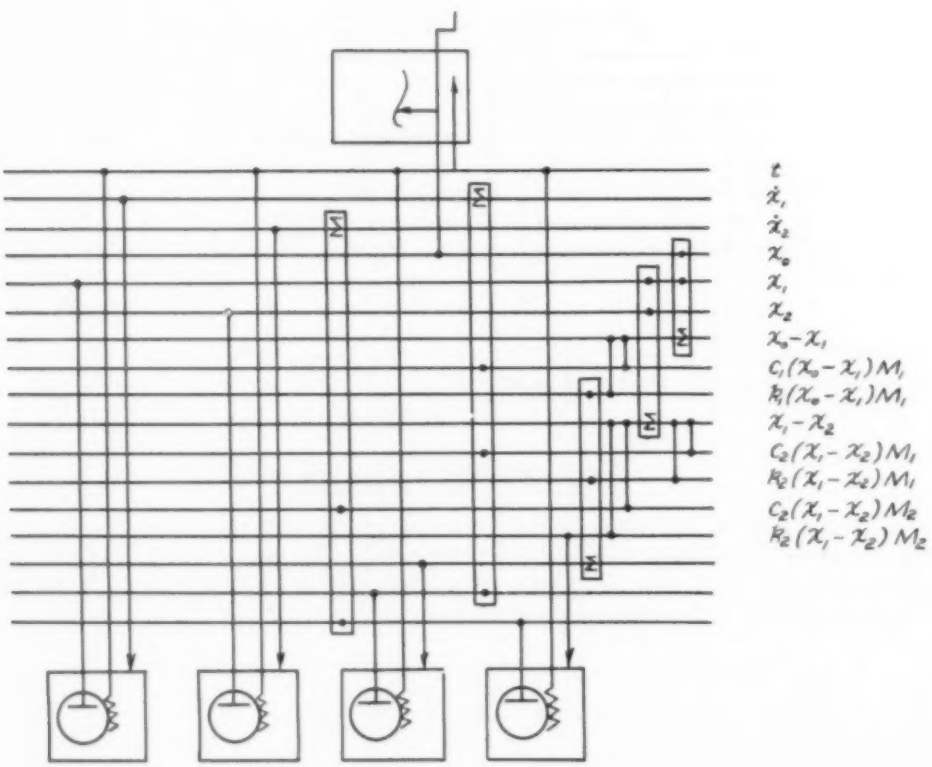
Figure 9

Dividing each equation by the mass and integrating once, we get

$$\dot{x}_1 = \frac{C_1}{M_1}(x_o - x_1) + \frac{C_2}{M_1}(x_2 - x_1) + \int \left[\frac{k_1}{M_1}(x_o - x_1) + \frac{k_2}{M_1}(x_2 - x_1)\right] dt$$

$$\dot{x}_2 = \frac{C_2}{M_2}(x_1 - x_2) + \frac{k_2}{M_2}\int (x_1 - x_2)\, dt.$$

The differential analyzer layout is the following



CONCLUSION

In recent years new methods in the art of solving problems on the differential analyzer have been developed. Notable among these is the method of inverse integration without the use of servomechanisms developed by J. G. L. Michel. (See bibliography reference at the end of the paper.) The application of the differential analyzer to the solution of partial differential equations with special boundary conditions has been made by D. R. Hartree.

The differential analyzer is a powerful tool in the solution of differential equations. It can, of course, be used in problems for which the solution can be obtained by analytical methods. However, if a large number of solutions are desired for different values of the parameters involved, the machine method is the only practical one. The same is true if the equation if non-linear or if experimental data are to be used.

The differential analyzer, since its installation at Wayne University, has been used to solve a number of important problems dealing with the automobile. Likewise, its application to the determination of the protein fractions of the blood to ascertain the presence of disease has been noteworthy.

Its proper utilization for training purposes, for research work in the University and by local industry in the solution of basic engineering problems can materially advance the methods of research and bring about fundamental improvements in design and greater economy and better quality of products.

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FLUID FLOW MAPPERS *

A. D. Moore

The fields in electromagnetic, heat flow, chemical diffusion, and other problems may be simulated by the flow of fluids in suitable channels. Despite the pioneering work of Hele-Shaw in 1901[1] little work has been done in this field up to now, because of the experimental difficulties of his method. However, if modern dental stone is cast in the form required, using a sheet of plate glass as a mold, it is possible to obtain the uniform flow passage essential for accurate work.[2]

The major components of a fluid flow mapper are a plaster slab and a piece of plate glass, parallel to the slab and spaced upward from it by a small constant distance. This creates the necessary two dimensional flow space. The combination is immersed in a tray of water, and small tanks connected by rubber tubing to holes in the slab act as sources and sinks to create the flow. The flow is made visible by crystals of potassium permanganate and barriers are added as needed to represent the boundary conditions of the field.

After the slab has been made by the methods detailed in reference 2, the fields are made visible by flowing water over a sprinkling of colored crystals. The flow patterns may be recorded photographically and are then available for detailed computation of potential lines from the streamlines recorded. These pictures are intrinsically interesting as patterns and color photographs of great beauty have been made.

Distributed sources of flows represent the magnetic field in a region carrying current and may be simulated by a "sand bed"[3][4]. This technique can be used also to represent the solutions for the torsion of a prismatic beam and the flow of heat in a region where heat is generated uniformly.

The equipment has been used to solve a variety of practical problems, as a teaching aid, and to verify some theoretical propositions relative to electromagnetic fields.[4]

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* This is an abstract of a demonstration lecture given before the Society, December 2, 1949. Complete articles by the author on the subject are listed in the bibliography. Arthur Dearth Moore is Professor of Electrical Engineering at the University of Michigan.

A LOCATION ESTIMATOR FOR NON-GAUSSIAN DISTRIBUTION*

D. R. Elliott

Abstract

An expression has been found for an estimator of the location parameter of a non-Gaussian distribution. The estimator (g) satisfies the conditions (1) $g(X_1, X_2, \ldots X_n) = g(X_1+U, X_2+U, \ldots X_n+U) - U$ and (2) variance of (g) = a minimum where $X_1 \ldots X_n$ are the (n) sample variates ordered according to magnitude and not, in general, just any arbitrary ordering, and U is a parameter of location. Both the median and mean satisfy equation (1) but not necessarily equation (2). (g) for a Gaussian distribution gives g = the mean; (g) for a rectangular distribution gives g = (smallest variate plus largest variate)/2; more complicated expressions are obtained for Cauchy distribution. For some distributions, the use of the above estimator may be considerably more efficient than the average of the sample.

Sometimes in product control work, one knows from past experience the form of the distribution of a statistical variable but wishes to determine, by drawing a sample of (n) variates, whether the distribution has changed its location (shifted to higher or lower values) or is the same. It is customary in such cases to determine the average of the (n) sample variates and to plot the average on a chart along with previously obtained values, using time as the abscissa. If the distribution does not change in time, the successive points on the chart fluctuate about a horizontal straight line. If the distribution shifts steadily with time, then the successive points fluctuate about a straight line having a slope. Since the basic reason for taking a sample is to determine whether the distribution has shifted, it is desirable that the fluctuations of the points on the chart be small in order to detect any shift as soon as possible. If one is limited in the number of variates at his disposal, he may desire to use a more accurate method of determining the location of the distribution than to form the average, even if the calculation is more time consuming. The method of calculation presented here may not always be the most efficient method but is more accurate than taking the average, except in those cases that it reduces to taking the average.

It seems obvious that any function (g) of the variates capable of detecting a shift in the distribution should satisfy the condition

$$\overline{g(X_1, X_2, \ldots X_n)} = \overline{g(X_1+U, X_2+U, \ldots X_n+U) - U} \qquad (1)$$

Equation (1) states that if the distribution shifts by an amount U, then the average value of an estimator (g) must also increase by an amount U. One solution of equation (1) is

$$g(X_1, X_2, \ldots X_n) = g(X_1+U, X_2+U, \ldots X_n+U) - U \qquad (2)$$

Although equation (2) is not the most general solution of equation (1), it is sufficiently general to include the average and in many cases, estimators which are considerably more accurate than the average.

As a gauge of accuracy, we use the condition that of all possible solutions of equation (2), the solution having the least variance is the best. Hence, if (p) represents

*A paper presented at the joint session of the American Mathematical Society and the Industrial Mathematics Society at East Lansing on February 25, 1950. Mr. Elliott is with the United States Rubber Company, Detroit, Michigan.

the joint distribution function of the variates $X_1, X_2, \ldots X_n$

$$\int g^2 p \, dx_1, dx_2 \ldots dx_n = \text{a minimum} \tag{3}$$

where the integration extends over all values of $X_1, X_2, \ldots X_n$, but (p) may be zero outside a finite range.

Equation (3) states that the variance of (g) is to be a minimum.

The solution of equations (2) and (3) is easily obtained (see Appendix) and is

$$g(X_1, X_2, \ldots X_n) = \frac{-\int X p(X+X_1, X+X_2, \ldots X+X_n)dx}{\int p(X+X_1, X+X_2, \ldots X+X_n)dx} \tag{4}$$

It is not necessary that the joint distribution function $p(X_1, X_2, \ldots X_n)$ be expressable in the form

$$p = \phi(X_1)\,\phi(X_2)\ldots\phi(X_n) \tag{5}$$

However, if (p) is a symmetric function of the variables as in equation (5), (g) is also a symmetric function. Moreover, if (p) is given explicitly with a parameter of location θ in the form

$$p = p(X_1-\theta,\ X_2-\theta, \ldots X_n-\theta) \tag{6}$$

then (g) is an unbiased estimate of the parameter θ.

If the distribution is Gaussian, then

$$p = A^n e^{-1/2 (X_1^2 + X_2^2 + \ldots X_n^2)}$$

and

$$g = \frac{-\int_{-\infty}^{\infty} X e^{-1/2\left(\left[X+X_1\right]^2 + \left[X+X_2\right]^2 + \ldots \left[X+X_n\right]^2\right)} dx}{\int_{-\infty}^{\infty} e^{-1/2\left(\left[X+X_1\right]^2 + \left[X+X_2\right]^2 + \ldots \left[X+X_n\right]^2\right)} dx}$$

$$g = \frac{-\int_{-\infty}^{\infty} X e^{-1/2\left(nX^2+2X\left[X_1+X_2+\ldots X_n\right] + X_1^2+X_2^2+\ldots X_n^2\right)} dx}{\int_{-\infty}^{\infty} e^{-1/2\left(nX^2+2X\left[X_1+X_2+\ldots X_n\right] + X_1^2+X_2^2+\ldots X_n^2\right)} dx}$$

Multiply numerator and denominator by proper factor.

$$g = \frac{-\int_{-\infty}^{\infty} X e^{-\frac{n}{2}\left[X+\bar{X}\right]^2} dx}{\int_{-\infty}^{\infty} e^{-\frac{n}{2}\left[X+\bar{X}\right]^2} dx} \qquad \text{where } \bar{X} = \frac{X_1+X_2\ldots X_n}{n}$$

Let $y = X + \bar{X}$

$dy = dx$

$$g = \frac{-\int_{-\infty}^{\infty} (y-\bar{X}) e^{-\frac{n}{2}y^2}\, dy}{\int_{-\infty}^{\infty} e^{-\frac{n}{2}y^2}\, dy} = \bar{X}$$

$$g = \frac{X_1+X_2+\ldots X_n}{n}$$

Thus in the case of a Gaussian distribution, the function (g) reduces to the average and no improvement over the usual practice is found.

If the distribution is rectangular, then

$$p = A^{n}\, S(X_1)\, S(X_2) \ldots S(X_n)$$

where $S(X) = 1 \qquad |X| \leq a$ (7)

and $S(X) = 0 \qquad |X| > a$ (7)

and A is the normalizing factor.

Therefore, from equation (4)

$$g = \frac{-\int_{-\infty}^{\infty} X\, S(X+X_1)\, S(X+X_2) \ldots S(X+X_n)\, dx}{\int_{-\infty}^{\infty} S(X+X_1)\, S(X+X_2) \ldots S(X+X_n)\, dx} \tag{8}$$

Let X_L = the variate having the lowest value and X_h = the variate having the highest value.

Because of equation (7), the integrands vanish for

$X+X_L < -a$ and $X+X_h > a$

or for $X < -a-X_L$ and $X < a-X_h$ (9)

Hence, equation (9) becomes

$$g = \frac{-\int_{-a-X_L}^{a-X_h} X\, dx}{\int_{-a-X_L}^{a-X_h} dx} \qquad \text{or} \tag{10}$$

$$g = \frac{X_h+X_L}{2}$$

Equation (10) shows that for a rectangular distribution, the average of the highest and lowest variate is the improved method of calculation.

For some distributions, equation (4) yields expressions considerably more complicated. Hence, for Cauchy's distribution

$$p = \frac{A^n}{(1+X_1^2)(1+X_2^2)\ldots(1+X_n^2)}$$

gives for

$$n = 2 \qquad g_2 = \frac{X_1+X_2}{2}$$

and for $n = 3$

$$g = \frac{8(X_1+X_2+X_3) - 6X_1X_2X_3+X_1^2(X_2+X_3) +X_2^2(X_1+X_3) +X_3^2(X_1+X_2)}{2(X_1^2+X_2^2+X_3^2 -X_1X_2 -X_1X_3 -X_2X_3 +12)}$$

APPENDIX

Given $g(X_1, X_2, \ldots X_n) = g(X_1+U, X_2+U, \ldots X_n+U) - U$ (1)

and $I = \int_{-\infty}^{\infty} g^2 p\, dx_1\, dx_2 \ldots$ (2)

where $p = p(X_1, X_2, \ldots X_n)$

To find the function (g) which satisfies equation (1) and which minimizes the right side of equation (2) transform equations (1) and (2) to the new set of variables given as follows:

$$X = X_1$$

$$y_1 = X_2 - X_1 \qquad (3)$$

$$y2 = X_3 - X_1$$

$$y_{n-1} = X_n - X_1$$

Equation (1) becomes

$$F(X, y_1, y_2 \ldots y_{n-1}) = F(X+U, y_1, y_2 \ldots y_{n-1}) - U \qquad (4)$$

$$\text{where } F(X, y_1, y_2 \ldots y_{n-1}) = g(X_1, X_2 \ldots X_n) \qquad (5)$$

Equation (2) becomes

$$I = \int_{-\infty}^{\infty} F^2 \phi\, dx\, dy_1\, dy_2 \ldots dy_{n-1} \qquad (6)$$

$$\text{where } \phi(X, y_1, y_2, \ldots y_{n-1}) = p(X_1, X_2, \ldots X_n) \qquad (7)$$

Since equation (4) must hold for all values of U, if we expand the right member of equation (4) into a Taylor series in U about the point $X, y_1, y_2 \ldots y_{n-1}$, we obtain

$$\frac{\partial F}{\partial X} = 1 \tag{8}$$

$$\text{or} \quad F = X+f(y_1, y_2, \ldots y_{n-1}) \tag{9}$$

where (f) is any arbitrary function of the y's. Since equation (9) was a necessary condition on the solution of equation (4) and substitution into equation (4) shows it to be a sufficient condition, equation (9) is the general solution of equation (1).

For (I) given by equation (6) to be a minimum, it is necessary that $\delta I = 0$

$$\text{or} \quad \delta I = 2\int F \phi \, \delta F \, dx \, dy_1 \, dy_2 \ldots dy_{n-1} = 0 \tag{10}$$

for all δF compatible with equation (9).

But by equation (9), the form of F is determined except for the form of (f), hence

$$\delta F = \delta f \tag{11}$$

Substituting equations (11) and (9) into equation (10) yields

$$\delta I = 2\int (X+f) \phi \, \delta f \, dx \, dy_1 \, dy_2 \ldots dy_{n-1} = 0 \tag{12}$$

Since δf is a function not dependent on X, equation (12) may be written

$$2\int \delta f \, dy_1 \, dy_2 \ldots dy_{n-1} \int (X+f) \phi \, dx = 0 \tag{13}$$

Equation (13) may be satisfied for arbitrary (δf) only if

$$\int (X+f) \phi \, dx = 0$$

$$\text{or} \quad f = \frac{-\int X \phi \, dx}{\int \phi \, dx} \tag{14}$$

$$\therefore F = X_1 \frac{-\int X \phi \, dx}{\int \phi \, dx} = \frac{-\int (X-X_1) \phi \, dx}{\int \phi \, dx} \tag{15}$$

If we transform the right side of equation (15) back into the old variables, we obtain

$$g = \frac{-\int_{-\infty}^{\infty} X p(X+X_1, X+X_2, \ldots X+X_n) \, dx}{\int_{-\infty}^{\infty} p(X+X_1, X+X_2, \ldots X+X_n) \, dx} \tag{16}$$

Q.E.D.

Since (g) given by equation (16) is the function of minimum variance, the function (g+a) must have greater variance than (g) for all values of the parameter (a) different from 0. This can be true only if

$$\bar{g} = 0 \tag{17}$$

Hence, it follows from equations (1) and (17) that if the distribution (p) is given explicitly with a location parameter θ

$$p = p(X_1-\theta, X_2-\theta, \ldots X_n-\theta)$$

Then (g) given by equation (16) is an unbiased estimate of the parameter θ.

ACKNOWLEDGMENTS

I wish to acknowledge the assistance and interest given to this work by Professor Benjamin Epstein of Wayne University and Mr. William Pangonis of the United States Rubber Company.

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The Differential Equations of Structures

Clayton J. Pajot [1]

Introduction

The following discussion is intended to show the derivation and application of some of the differential equations most commonly used in structural engineering.

The discussion is subdivided into the following headings:

1. Derivation of the fundamental equations relating to bending moment, stress, deflections and the beam constants.

2. Application of the fundamental equations to beams, columns and beam columns.

3. Derivation and discussion of the energy method of finding deflections (castigliano theorem).

4. Extension of the fundamental equations to certain cases of plates or slabs.

Assumptions

The assumptions made here are those usually made when discussing problems in elementary strength of materials, namely that:

1. The material is perfectly elastic (stress is proportional to strain).

2. The material is isotropic.

3. The principle of superposition holds.

4. The ratio of stress to strain is the same in tension and compression.

REFERENCES

Timoshenko - Strength of Materials - Vols. I and II
Timoshenko - Theory of Elastic Stability
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PART ONE

FIGURE 1A

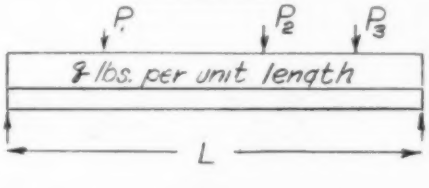
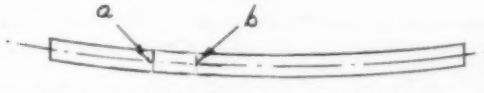


FIGURE 1B



Let Figure 1A represent a beam of length (L) and loaded as shown with concentrated loads P_1, P_2 and P_3, and a uniform load of (q) pounds per unit length.

[1] Orientation Lecture delivered before the Society, March 31, 1950. Clayton J. Pajot is Chief Stress Engineer, Nash Research Department, Nash-Kelvinator Corporation.

Figure 1B represents the deflected position of the beam in exaggerated form. Evidently the upper fibres of the beam are compressed or shortened, whereas the lower fibres are elongated. There is then some plane between the upper and lower surfaces of the beam in which the fibres are neither shortened nor elongated. This is called the neutral plane of the beam and is represented by the broken line passing through the points (a) and (b) in Figure 1B.

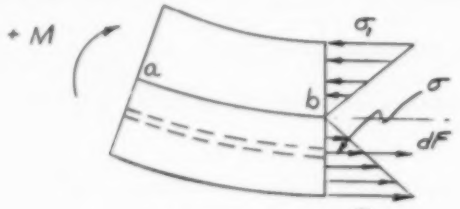


FIGURE 2A

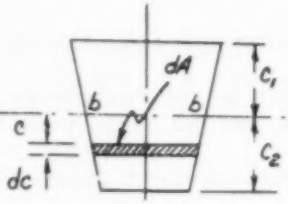


FIGURE 2B

Figure 2A represents an enlarged view of the element a - b. The external moment is shown to the left of the figure and represents the moment obtained from the ordinary moment diagram for the beam. It is assumed to be positive when it causes the beam to be concave upwards as shown. The triangles on the face (b) represent the variation in stress on the internal fibres of the beam. The stress is assumed to be proportional to the deformation and therefore has a zero value at the neutral plane a - b, a maximum compressive value σ_1 at the top and a maximum tensile value σ_2 at the bottom. The stress at any distance (c) from the neutral axis is represented by σ.

Figure 2B shows a section of the beam at the point (b). For this discussion it could be of any shape symmetrical to the vertical axis. For ease in drawing, it is assumed as shown. The line b - b is the intersection of the neutral plane with the vertical section and is called the neutral axis of the section.

(dF) is the total force due to fibre stress acting on the differential area dA.

THUS $dF = \sigma \cdot dA$

FROM SIMILAR TRIANGLES $\frac{\sigma}{\sigma_2} = \frac{c}{c_2}$ OR $\sigma = \frac{\sigma_2}{c_2} c$

$\therefore \quad dF = \frac{\sigma_2}{c_2} c\, dA$

AND

$$F = \int_{c_1}^{c_2} dF = \frac{\sigma_2}{c_2} \int_{c_1}^{c_2} c\, dA = \frac{\sigma_2}{c_2} \bar{c} A \qquad (1)$$

Now (F) represents the total force acting on the face (b) of the element. Since the element is in equilibrium due to moment (M) on the face (a) and the forces on the face (b), and since (M) represents a couple which has no resultant force it follows that

$$\int_{c_1}^{c_2} dF = 0 \quad \text{OR} \quad \frac{\sigma_2}{c_2} \bar{c} A = 0$$

BUT $\sigma \neq 0 \quad A \neq 0 \quad$ AND $\quad c_2 \neq \infty$

therefore $\bar{c} = 0$ and b - b is the centroidal axis of the section.

Let (dM) be the moment due to the differential force (dF). Then the external moment (M) is equal to the total moment due to the differential forces along the face (b).

OR $$M = \int_{c_1}^{c_2} dM = \int_{c_1}^{c_2} c\,dF = \frac{\sigma_2}{c_2}\int_{c_1}^{c_2} c^2 dA$$

$$I = \int_{c_1}^{c_2} c^2 dA$$

where (I) is the moment of inertia of the area of the section about axis b - b.

THEN $$M = \frac{\sigma_2}{c_2} I = \frac{\sigma}{c} I$$

AND $$\sigma = \frac{MC}{I} \qquad (2)$$

In Figure 3 is shown the element a - b of the beam in its deflected position. (o') is the center of curvature and ρ the radius of curvature of the plane a - b.

The line (bf) is drawn parallel to (ad). Then (df) is very nearly equal to (ab) and (fg) is the elongation of an element of original length (ab).

Let ε = unit strain at distance (C) from the neutral axis

then $$\varepsilon = \frac{fg}{df} = \frac{fg}{ab}$$

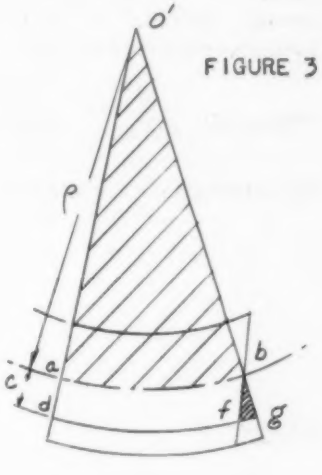


FIGURE 3

considering the similar areas (bfg) and (abo')

$$\frac{fg}{ab} = \frac{c}{\rho} \quad \text{OR} \quad \varepsilon = \frac{c}{\rho}$$

Let (E) = the modulus of elasticity of the material

$$E = \frac{\text{STRESS}}{\text{STRAIN}} = \frac{\sigma}{\varepsilon} \quad \text{OR} \quad \varepsilon = \frac{\sigma}{E}$$

THEN $$\frac{\sigma}{E} = \frac{c}{\rho} \quad \text{OR} \quad \sigma = \frac{EC}{\rho}$$

BUT FROM (2) $$\sigma = \frac{MC}{I}$$

$$\therefore \quad \frac{EC}{\rho} = \frac{MC}{I} \quad \text{AND} \quad \frac{1}{\rho} = \frac{M}{EI} \qquad (3)$$

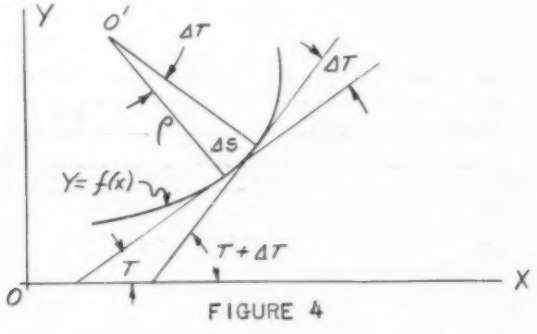


FIGURE 4

Figure 4 represents any curve Y = f(x) whose curvature $\frac{1}{\rho} = 1/\rho$ is to be determined. From the figure $\Delta s \doteq \Delta t$ (approximately) and $1/\rho = \Delta t/\Delta s$ (approximately).

THEREFORE $K = \frac{1}{\rho} = \lim_{\Delta S \to 0} \frac{\Delta T}{\Delta S} = \frac{dT}{dS}$

IN FIGURE 4 $\frac{dY}{dX} = \tan T$ OR $T = \tan^{-1} \frac{dY}{dX}$

$$\frac{dT}{dx} = \frac{\frac{d^2Y}{dx^2}}{1 + \left(\frac{dY}{dx}\right)^2}$$, ALSO $\frac{dS}{dx} = \left[1 + \left(\frac{dY}{dx}\right)^2\right]^{1/2}$

AND $$\frac{dT}{dS} = \frac{\frac{dT}{dx}}{\frac{dS}{dx}} = \frac{\frac{d^2Y}{dx^2}}{\left[1 + \left(\frac{dY}{dx}\right)^2\right]^{\frac{3}{2}}} = \frac{1}{\rho}$$

OR $$\frac{M}{EI} = \frac{\frac{d^2Y}{dx^2}}{\left[1 + \left(\frac{dY}{dx}\right)^2\right]^{3/2}} \qquad (4)$$

Since, in the case of beams, the slope of the neutral plane at any point is in general small, $(dY/dx)^2$ in (4) may be disregarded when the formula is applied to members having small deflections.

Then (4) $$\frac{M}{EI} = \frac{d^2Y}{dx^2} \qquad (5)$$

which might be called the fundamental differential equation of structures.

PART TWO

(A) Beams

Let Figure 5 represent a beam having a uniform load of (q) pounds per unit length and the concentrated loads P_1 and P_2 and having end reactions of R_1 and R_2. The differential equations of deflection will be written for the beam.

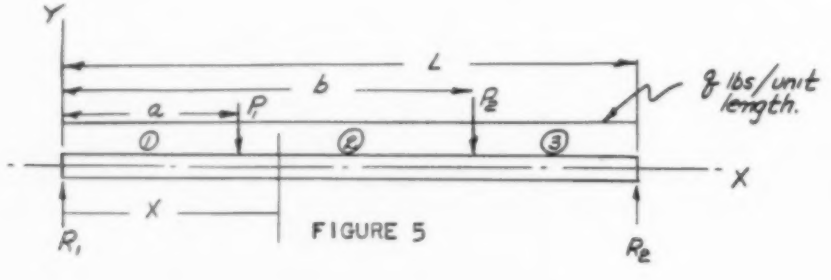


FIGURE 5

Note that the concentrated loads constitute points of discontinuity for the moment equations and that therefore three separate equations must be written, one for interval ①, one for interval ② and one for interval ③

The moment equations for the three intervals are:

$$M_1 = R_1 x - \frac{q x^2}{2} \qquad 0 < x < a$$

$$M_2 = R_1 x - \frac{q x^2}{2} - P_1(x-a) \qquad a < x < b$$

$$M_3 = R_1 x - \frac{q x^2}{2} - P_1(x-a) - P_2(x-b) \qquad b < x < L$$

Using equation (5), the corresponding differential equations are:

$$\frac{d^2Y}{dx^2} = \frac{M_1}{EI} = \frac{1}{EI}\left[R_1 x - \frac{q x^2}{2}\right] \qquad 0<x<a \quad (6)$$

$$\frac{d^2Y}{dx^2} = \frac{M_2}{EI} = \frac{1}{EI}\left[R_1 x - \frac{q x^2}{2} - P_1(x-a)\right] \qquad a<x<b \quad (7)$$

$$\frac{d^2Y}{dx^2} = \frac{M_3}{EI} = \frac{1}{EI}\left[R_1 x - \frac{q x^2}{2} - P_1(x-a) - P_2(x-b)\right] \qquad b<x<L \quad (8)$$

Note that if the cross section of the beam is not uniform throughout that the moment of inertia (I) will not be constant. This condition may be expressed as

$$I = F(x)$$

(B) Columns

Let Figure 6A represent a column fixed at its lower end and with an eccentric load (P) at a distance (e) from the center line of the column. Figure 6B represents the deflection curve of the neutral axis of the column and Figure 6C shows the deflection curve rotated to conform to the coordinate system as previously chosen.

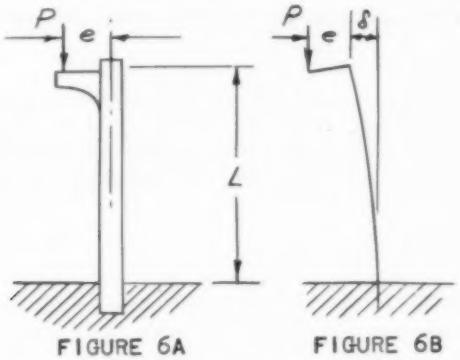


FIGURE 6A

FIGURE 6B

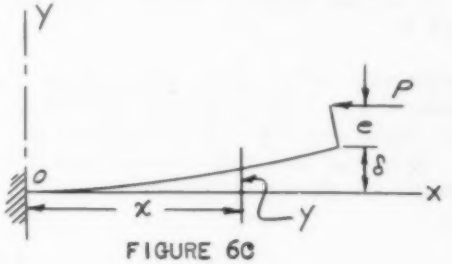


FIGURE 6C

From Figure 6C the moment equation is:

$$M = P(e+\delta-Y)$$

Then, from equation (5)

$$\frac{d^2Y}{dx^2} = \frac{M}{EI} = \frac{P}{EI}(e+\delta-Y) = K^2(e+\delta-Y)$$

WHERE $K^2 = \frac{P}{EI}$

OR $$\frac{d^2Y}{dx^2} + K^2 Y = K^2(e+\delta) \qquad (9)$$

The solution for (9) is

$$\delta = e\left(\sec L\sqrt{\frac{P}{EI}} - 1\right) \qquad (10)$$

Equation (10) is plotted in Figure 7.

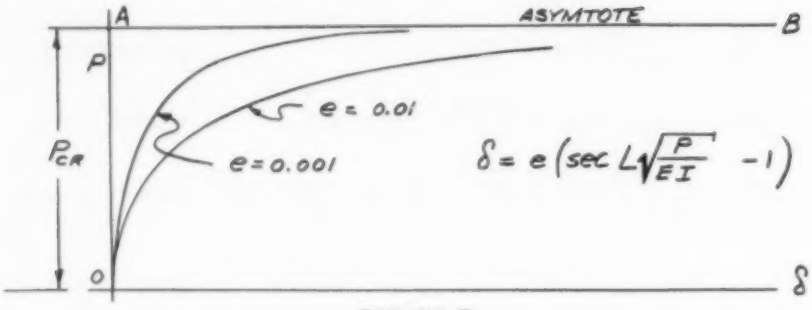


FIGURE 7

If there is no eccentricity, $e \longrightarrow o$. Then the so-called critical or buckling load is P_{cr} where $P_{cr} = \lim_{e \longrightarrow o.} P$.

The value of P_{cr} is given by

$$P_{CR} = \frac{\pi^2 EI}{4L^2} \qquad (11)$$

This is called Euler's column formula and corresponds to the condition shown in Figure 8.

Theoretically, when a concentric load (P) is placed upon the column and then gradually increased, the column should deflect in accordance with the curve oAB of Figure 7 which means that when the critical value of (P) is reached the column should suddenly fail. In actual test this failure does not occur, but when P_{cr} is reached, it deflects somewhat as shown in Figure 8.

FIGURE 8

The reason for this discrepancy can be attributed to the fact that when this condition occurs, we are dealing with large deflections but equation (6), upon which our solution is based, is to be used for small deflections only.

For large deflections, equation (4) should be used. Applying this equation to Figure 6C we have

$$\frac{\frac{d^2Y}{dx^2}}{\left[1+\left(\frac{dY}{dx}\right)^2\right]} = \frac{M}{EI} = \frac{P}{EI}(e+\delta-Y) \qquad (12)$$

In the case of a column with hinged ends, the solution of this equation has been found to be

$$\delta = \frac{L\sqrt{8}}{\pi}\sqrt{\frac{P}{P_{CR}}-1}\left[1-\frac{L}{8}\left(\frac{P}{P_{CR}}-1\right)\right] \qquad (13)$$

Figure 9 shows the graph of equation (13). It will be noted from this graph that when loads above the critical are reached that large deflections occur with small increase in load.

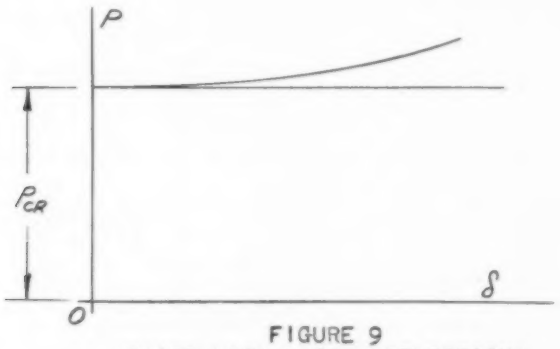


FIGURE 9
BASED UPON LARGE DEFLECTIONS

FIGURE 10

Figure 10 shows the manner in which a column having hinged or pinned ends deflects. This type is usually called the fundamental case. The critical load for this type, based upon the equation for small deflections, is

$$P_{CR} = \frac{\pi^2 E I}{L^2} \qquad (14)$$

A condition which sometimes occurs in practice is that shown in Figure 11A in which an initial eccentricity (e) exists before load is applied. The column deflects into the form 11B with a deflection δ as load is applied.

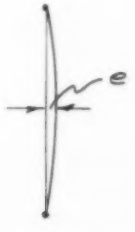


FIGURE 11A

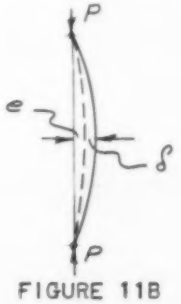


FIGURE 11B

FIGURE 12

Another type of column sometimes encountered in mechanical and aeronautical engineering work is the stepped column shown in Figure 12.

An interesting discussion of both types may be found in the October and November issues of Product Engineering magazine for the year 1949. The article is entitled "Uniform and Stepped Columns" and is by J. H. Meier.

Note that in all this discussion of columns it is assumed that we are dealing with long columns; that is, columns having cross-sectional dimensions that are small as compared to the length of the column. These are sometimes referred to as slender columns.

(C) Beam Columns

Beam columns occur frequently in aeronautical work. A beam column is simply a member having some form of transverse load in addition to its axial load.

Figure 13. shows a beam column having only one transverse load. It is subdivided into two intervals or parts numbered as (1) and (2) on each side of the transverse load, as the load constitutes a point of discontinuity for the moment equations. Two equations are then developed, one for each part.

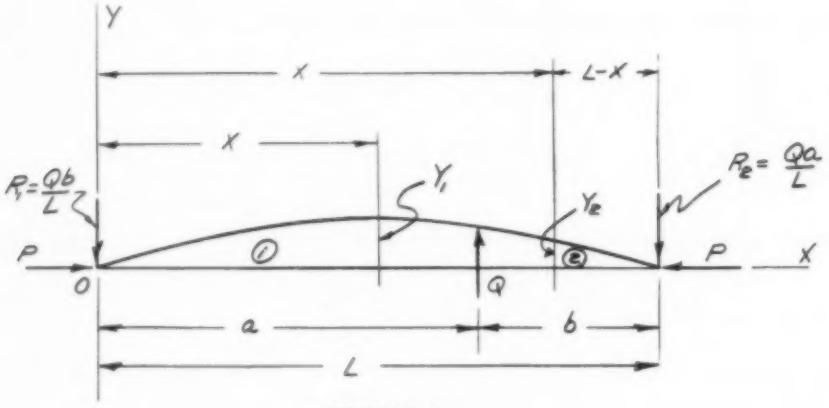


FIGURE 13

$$M_1 = -\frac{Qbx}{L} - PY_1 \quad , \qquad M_2 = \frac{Qa(L-x)}{L} - PY_2$$

FOR FIRST INTERVAL $0 < x < a$

$$\frac{d^2Y}{dx^2} = \frac{M_1}{EI} = -\frac{Qbx}{EIL} - \frac{PY}{EI}$$

LETTING $\frac{P}{EI} = K^2$

$$\frac{d^2Y}{dx^2} + K^2Y = -\frac{Qb}{EIL}x \qquad (15)$$

FOR SECOND INTERVAL $a < x < L$

$$\frac{d^2Y}{dx^2} = \frac{M_2}{EI} = -\frac{Qa(L-x)}{EIL} - \frac{PY}{EI}$$

AND LETTING $\frac{P}{EI} = K^2$

$$\frac{d^2Y}{dx^2} + K^2Y = -\frac{Qa}{EIL}(L-x) \qquad (16)$$

THE GENERAL SOLUTION OF (15) IS

$$Y_1 = A \sin Kx + B \cos Kx - \frac{Qbx}{PL} \qquad (17)$$

Where A sim Kx + B cos Kx is the complementary function and $-\frac{Qbx}{PL}$ is a particular solution.

Similarly, the general solution of (16) is

$$Y_2 = C \sin Kx + D \cos Kx - \frac{Qa(L-x)}{PL} \qquad (18)$$

SINCE $Y_1 = 0$ WHEN $x = 0$

$Y_2 = 0$ WHEN $x = L$

AND $Y_1 = Y_2$ AND $\frac{dY_1}{dx} = \frac{dY_2}{dx}$ WHEN $x = a$

The constants A, B, C and D may be evaluated.

$$A = \frac{Q \sin Kb}{PK \sin KL}, \quad B = 0, \quad C = -\frac{Q \sin Ka}{PK \tan KL}$$

AND $$D = \frac{Q \sin Ka}{PK}$$

Then, substituting these values in (17) and (18)

$$Y_1 = \frac{Q \sin Kb}{PK \sin KL} \sin Kx - \frac{Qbx}{PL} \qquad 0 < x < a \qquad (19)$$

$$Y_2 = -\frac{Q \sin Ka}{PK \tan KL} \sin Kx + \frac{Q \sin Ka}{PK} \cos Kx - \frac{Qa(L-x)}{PL} \qquad a < x < b \qquad (20)$$

Substituting the values of Y_1 and Y_2 in the moment equations:

$$M_1 = -\frac{Qbx}{L} - \frac{Q \sin Kb}{K \sin KL} \sin Kx + \frac{Qbx}{L} \qquad 0 < x < a \qquad (21)$$

$$M_2 = -\frac{Qa(L-x)}{L} + \frac{Q \sin Ka}{K \tan KL} \sin Kx - \frac{Q \sin Ka}{K} \cos Kx + \frac{Qa(L-x)}{L} \qquad b < x < L \qquad (22)$$

Ymax may be determined from (19) and Mmax from (21) or (22). This method may be extended to a condition having several concentrated loads. Note that

$$K = \sqrt{\frac{P}{EI}}$$

Another type of beam column is shown in Figure 14. In this case, in addition to the axial load, a uniformly distributed load of (q) pounds per unit length is applied, a moment or couple M_1 at the left end and a moment M_2 at the right end.

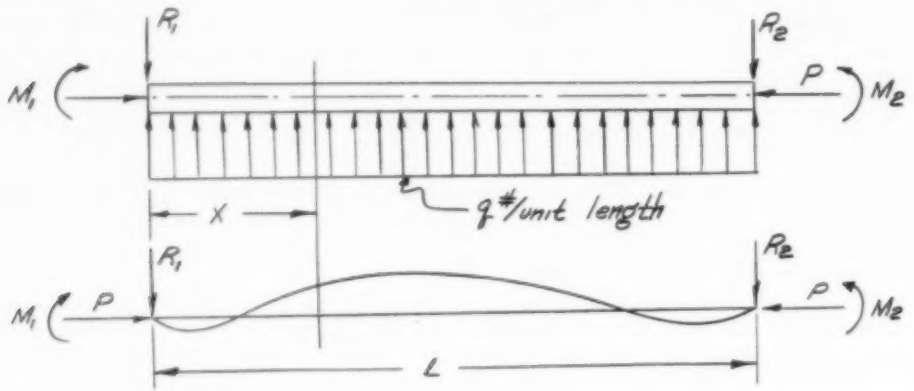


FIGURE 14

The method of solution in this case is slightly different than in the previous case. Although a differential equation of the same form is derived.

$$R_1 = \frac{qL}{2} + \frac{M_1 - M_2}{L}, \qquad R_2 = \frac{qL}{2} + \frac{M_2 - M_1}{L}$$

$$M = M_1 - R_1 X - PY + \frac{qX^2}{2} \tag{23}$$

$$\frac{dM}{dX} = -R_1 - P\frac{dY}{dX} + qX$$

$$\frac{d^2M}{dX^2} = -P\frac{d^2Y}{dX^2} + q$$

BUT FROM (5) $\quad \dfrac{d^2Y}{dX^2} = \dfrac{M}{EI}$

AND THEREFORE $\quad \dfrac{d^2M}{dX^2} + \dfrac{PM}{EI} = q$

LET $\quad \dfrac{P}{EI} = K^2$

THEN $\quad \dfrac{d^2M}{dX^2} + K^2M = q \tag{24}$

THE GENERAL SOLUTION IS

$$M = A \sin KX + B \cos KX + \frac{q}{K^2} \tag{25}$$

where A sin Kx + B cos Kx is the complementary function and q/K^2 is a particular solution.

WHERE $A \sin KX + B \cos KX$ IS THE COMPLEMENTARY FUNCTION AND q/K^2 IS A PARTICULAR SOLUTION.

WHEN $X = 0$, $M = M_1$ AND WHEN $X = L$, $M = M_2$

$$\therefore \quad A = \frac{M_2 - \frac{q}{K^2} - \left(M_1 - \frac{q}{K^2}\right)\cos KL}{\sin KL} \quad \text{AND} \quad B = M_1 - \frac{q}{K^2}$$

Substituting the values for and in (25)

$$M = \frac{M_2 - \frac{q}{K^2} - \left(M_1 - \frac{q}{K^2}\right)\cos KL}{\sin KL} \sin Kx + \left(M_1 - \frac{q}{K^2}\right)\cos Kx + \frac{q}{K^2} \quad (26)$$

If there are no end moments $M_1 = M_2 = 0$, then

$$M = \frac{-\frac{q}{K^2} + \frac{q}{K^2}\cos KL}{\sin KL} \sin Kx - \frac{q}{K^2}\cos Kx + \frac{q}{K^2}$$

$$= \frac{q}{K^2}\left[\left(\frac{\cos KL - 1}{\sin KL}\right)\sin Kx - \cos Kx + 1\right] \quad (27)$$

Mmdx may be obtained from (26) or (27). The equation of the deflection curve may be obtained by substituting equation (26) in (23) and then solving for Y_2. This gives

$$Y = \frac{1}{D}\left[M_1 - R_1 x + \frac{qx^2}{2} - \frac{M_2 - \frac{q}{K^2} - \left(M_1 - \frac{q}{K^2}\right)\cos KL}{\sin KL} \sin Kx - \left(M_1 - \frac{q}{K^2}\right)\cos Kx - \frac{q}{K^2}\right] \quad (28)$$

The slope equation is obtained by differentiating (28). It is

$$\frac{dY}{dx} = \left[\frac{1}{D} - R_1 + qx - \frac{M_2 - \frac{q}{K^2} - \left(M_1 - \frac{q}{K^2}\right)\cos KL}{\sin KL} K\cos Kx + \left(M_1 - \frac{q}{K^2}\right) K \sin Kx\right] \quad (29)$$

Beam columns continuous over several supports may be solved by applying equations (26) and (29) to each span independently.

A value for either the slope or the moment must be known at the beginning of the first span and at the end of the last span. It must be remembered that the slope at the end of any given span is equal to the slope at the beginning of the adjacent span. Thus a series of simultaneous linear equations may be set up by means of which all the moments over supports can be obtained. Then, knowing the end moments for any span, the maximum moment for the span may be obtained from (26).

PART THREE

(A) Energy in Beam Based Upon Bending Moment

Let Figure 15 represent a beam of length (L) with concentrated loads and uniform loads as indicated. The element a - b is any element of differential length (dx) located at a distance (x) from the left end.

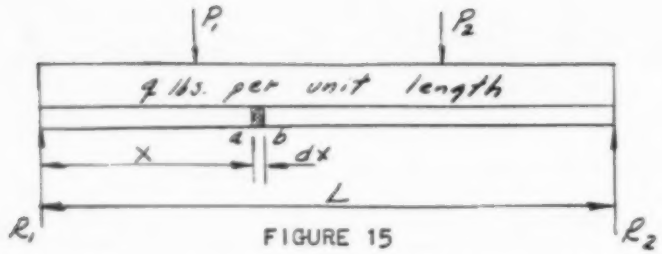


FIGURE 15

Let (M) represent the moment at point (x) on the beam, obtained from the usual beam moment diagram.

Figure 16 represents an enlarged view of the element a - b shown with the deflection due to the moment (M) exaggerated.

dU = work stored in element a - b due to bending

$$= \frac{1}{2} M\, d\phi$$

WHERE $d\phi$ = ANGLE OF BEND

FROM FIGURE 16,

$$dx = \rho\, d\phi \quad \text{OR} \quad \frac{1}{\rho} = \frac{d\phi}{dx}$$

USING EQUATION (3) $\quad \frac{1}{\rho} = \frac{M}{EI} = \frac{d\phi}{dx}$

AND $\quad d\phi = \frac{M}{EI}\, dx$

THEN $\quad dU = \frac{1}{2} M\, d\phi = \frac{1}{2} \frac{M^2}{EI}\, dx$

THEN THE TOTAL ENERGY OR WORK STORED IN THE BEAM IS GIVEN BY

$$U = \int dU = \int_0^L \frac{M^2 dx}{2EI} \qquad (30)$$

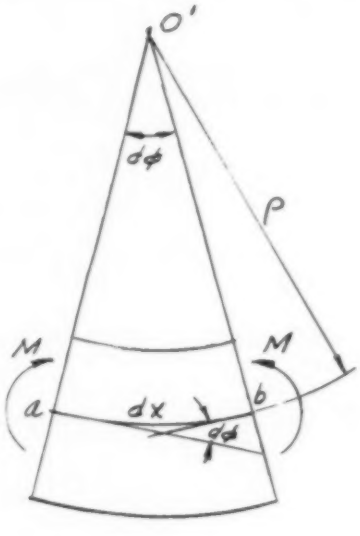


FIGURE 16

Then the total energy of work stored in the beam is given by

(B) Castigliano Theorem

Consider a beam loaded as shown in Figure 17.

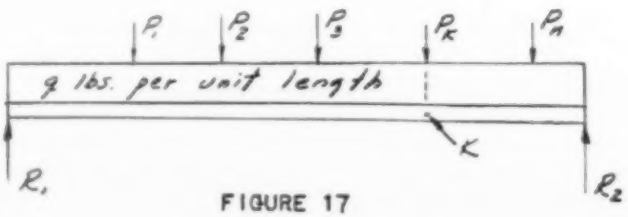


FIGURE 17

Let (U) be the energy stored in the beam due to the application of the loads shown. Then $U = f(q, P_1, P_2, P_3 \text{-----} P_k \text{----} P_n)$.

Suppose that at point K an additional differential load dP_k is placed (See Figure 18).

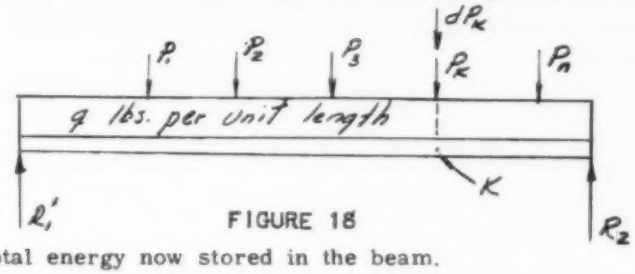


FIGURE 18

Let (U') be the total energy now stored in the beam.

$$U' = U + \frac{\partial U}{\partial P_k} dP_k$$

Now consider the case in which the differential load dP_k is first placed on the beam, and then the given load as shown in Figure 17 is superimposed. The total energy now stored in the beam is the same as in the first case and may be again represented by (U'). Thinking of (U') from the standpoint of this second procedure it may be represented as follows:

$$U' = U + (dP_k)\delta_k$$

where (U) and dP_k are defined as before, but δ_k is the deflection at (K) due to the application of the loading condition represented by Figure 17.

Since the two values of (U') are equal

$$U + \frac{\partial U}{\partial P_k} \cdot dP_k = U + (dP_k)\delta_k$$

$$\text{OR} \quad \frac{\partial U}{\partial P_k} \cdot dP_k = \delta_k \cdot dP_k \quad \text{OR} \quad \delta_k = \frac{\partial U}{\partial P_k} \qquad (31)$$

which is the theorem of castigliano.

From equations (30 and (31) an expression for the deflection may be written as follows:

$$\delta_k = \frac{\partial U}{\partial P_k} = \frac{\partial}{\partial P_k}\int_0^L \frac{M^2 dx}{2EI} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P_k} dx \qquad (32)$$

An expression similar to (32) may be written for curved members or beams, the requirement in its application being that the radius of curvature of the member is large as compared to the depth of the member. Figure 19 shows a curved beam or arch and equation (33) is the corresponding equation.

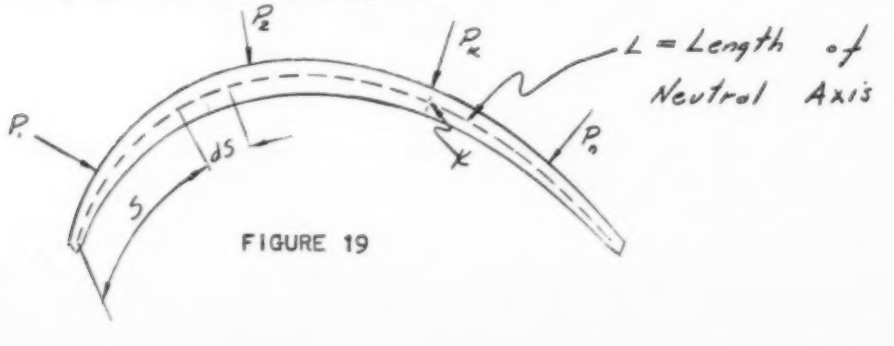


FIGURE 19

$$\delta_k = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial P_k} ds \qquad (33)$$

In equation (33) (ds) is a differential distance measured along the neutral axis and S_k is the deflection of the member at point (K) and in the direction P_k due to the given loading condition.

(C) Discussion

Equations (32) or (33) may be solved mathematically when (M) is expressed in terms of the known loading and (x) or (s), provided that the moment of inertia (I) is either a constant or can be expressed as a function of (x) or (s).

However, either equation (32) or (33) lends itself very well to graphical analysis when (I) cannot be expressed satisfactorily as a function of (x) or (s) or when the loading condition is such as to make it difficult to express the moment mathematically.

Although most of the equations that have been discussed here are related to deflections, deflections in themselves are not necessarily the most improtant thing to consider when dealing with engineering problems. In many cases the object of obtaining deflection equations is to use them in finding reactions, and ultimately moments and stresses in the indeterminate members or structures.

For example, in Figure 20A is shown a curved member represented by the curved line shown with both ends (A) and (B) fixed, and with the loads indicated.

Figure 20B shows the same structure but the fixed support at (A) has been replaced by the three reactions H_a, V_a, and M_a. In this figure the loads P_1, P_2, and P_3 are known, whereas the reactions are the unknowns to be solved for.

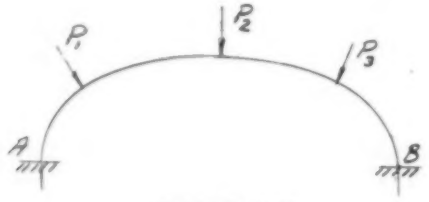


FIGURE 20A

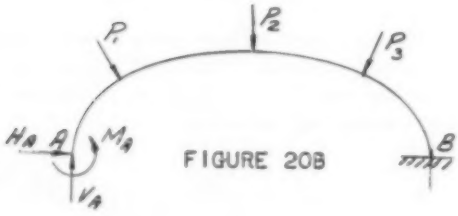


FIGURE 20B

Let M = f (H_a, V_a, M_a, loads) where (M) is the moment at any point such as (C) and is obtained by taking moments of all forces to the left of point (C).

From equation (33)

$$\delta_{H_A} = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial H_A} ds = 0 \qquad (34A)$$

where δ_h is the deflection of the end (A) in the direction H_a and is zero due to the fact that end (A) is fixed.

Similarly,

$$\delta_v = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial V_A} ds = 0 \qquad (34B)$$

where δ_v is the deflection of end (A) in the direction V_a and

$$\theta_A = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial M_A} ds = 0 \qquad (34C)$$

where θ_a is the rotation of the end (A).

Equation (34A) will be of the type

$$f_1(H_A, V_A, M_A, loads) = 0 \qquad (35A)$$

and similarly (35B) and (35C) will be of the form

$$f_2(H_A, V_A, M_A, loads) = 0 \qquad (35B)$$

$$f_3(H_A, V_A, M_A, loads) = 0 \qquad (35C)$$

Equations (35) are simultaneous linear equations that may be solved for the unknowns H_a, V_a, and M_a.

The reactions at (B) may then be determined from the equations of statics and also the maximum moment and maximum stress in the structure.

PART FOUR

(A) Rectangular Plates

Let Figure 21 represent a thin rectangular plate of thickness (h). M_x is the moment per unit length applied along the (Y) axis as shown and M_y is the moment per unit length applied along the (X) axis.

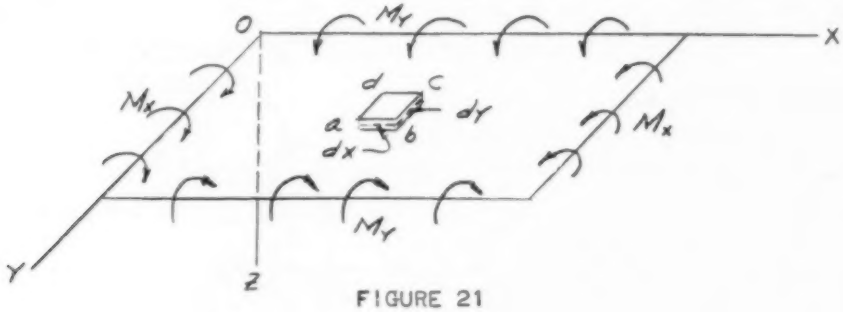


FIGURE 21

(abcd) is an element of the plate of dimensions dx and dy shown enlarged in Figure 22. Compare with Figures 2A and 2B.

Let σ_x be the stress per unit area on this element in the (x) direction and σ_y the stress per unit area in the (y) direction.

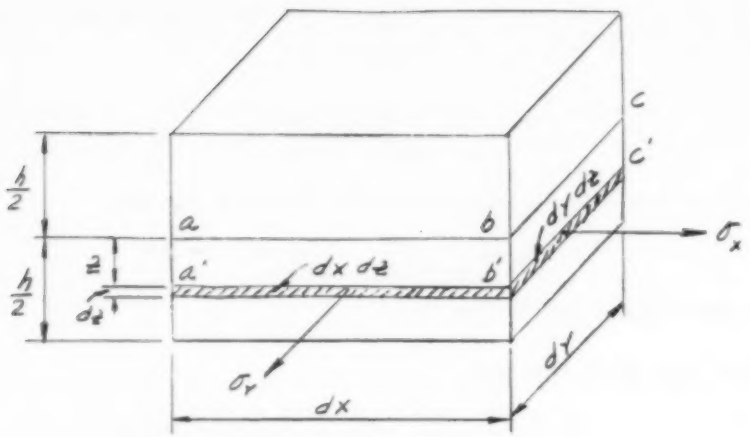


FIGURE 22

(a'b'c'd') is an element of Figure 22 of thickness d z and is shown separately in Figure 23 with curvatures due to the bending moment. The points (O') and (O") are the centers of curvature in the planes parallel to the x z and y z planes respectively.

$$\varepsilon_x = \frac{z}{\rho_x} \text{ AND } \varepsilon_y \frac{z}{\rho_y}$$

where ε_x and ε_y are the strains in the (x) and (y) directions respectively. Compare with derivation of $e = z/\rho$ from Figure 3.

ALSO $\varepsilon_x = \frac{1}{E}(\sigma_x - \mu\sigma_y)$

AND $\varepsilon_y = \frac{1}{E}(\sigma_y - \mu\sigma_x)$

where μ is Poisson's ratio.

Eliminating ε_x from the last two sets of equations;

$$\sigma_x = \frac{Ez}{1-\mu^2}\left(\frac{1}{\rho_x} + \mu\frac{1}{\rho_y}\right)$$

AND $$\sigma_y = \frac{Ez}{1-\mu^2}\left(\frac{1}{\rho_y} + \mu\frac{1}{\rho_x}\right)$$

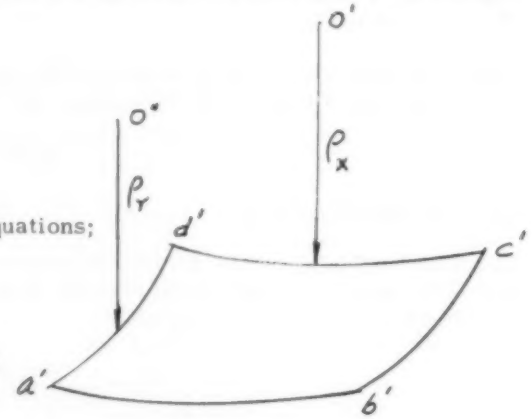


FIGURE 23

FROM FIGURES 21 AND 22.

$$M_x dy = \int_{-\frac{h}{2}}^{+\frac{h}{2}} z\sigma_x \, dy\, dz \qquad \text{OR} \qquad M_x = \int_{-\frac{h}{2}}^{+\frac{h}{2}} z\sigma_x \, dz$$

THEN $$M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{Ez^2}{1-\mu^2}\left(\frac{1}{\rho_x} + \mu\frac{1}{\rho_y}\right) dz$$

$$= \frac{E}{1-\mu^2}\left(\frac{1}{\rho_x} + \mu\frac{1}{\rho_y}\right)\int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 dz$$

$$= \frac{E}{1-\mu^2}\left(\frac{1}{\rho_x} + \mu\frac{1}{\rho_y}\right)\frac{h^3}{12}$$

LET $$\frac{Eh^3}{12(1-\mu^2)} = D \qquad (36)$$

THEN $$M_x = \left(\frac{1}{\rho_x} + \mu \frac{1}{\rho_y}\right) \qquad (37)$$

AND, SIMILARLY, $$M_y = D\left(\frac{1}{\rho_y} + \mu \frac{1}{\rho_x}\right) \qquad (38)$$

AS FOR EQUATION (5)

$$\frac{1}{\rho_x} = -\frac{\partial^2 w}{\partial x^2} \qquad \text{AND} \qquad \frac{1}{\rho_y} = -\frac{\partial^2 w}{\partial y^2}$$

where (w) is the deflection of the plate and the negative sign is introduced since the z axis is taken as positive downward.

THEN $$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) \qquad (39)$$

AND $$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right) \qquad (40)$$

Compare equations (39) and (40) with equation (5).

(B) Circular Plates

In Figure 24A is shown a thin circular plate of thickness (h). The loading on the plate is symmetrical to the center and is the same in any section through the center as shown in Figure 24C. (R) represents the reaction along the circumference of the plate and is in pounds per unit length. Figure 24B shows the curvature in any diametral section and also the radius of curvature ρ_1 in this section. Figure 24E is a diametral section but shows the radius of curvature ρ_2 in a direction perpendicular to the section (circumferential direction). Note that the center of curvature corresponding to ρ_2 is on the axis O z as elements such as the one through (A) form a conical surface with the apex at (O"). The moments M_1 and M_2 shown in Figures 24B and 24D are moments per unit length of direction ab or bc.

From Figure 24B: $$\phi = -\frac{dw}{dx}$$

Note that (w) is taken as positive downward hence the negative sign.

$$\frac{1}{\rho_1} = -\frac{d^2 w}{dx^2} = \frac{d\phi}{dx} \qquad \text{SEE EQUATIONS (3) AND (5)}$$

FROM FIGURE 24E: $$X = \rho_2 \sin\phi \approx \rho_2 \phi$$

OR $$\frac{1}{\rho_2} = \frac{\phi}{X}$$

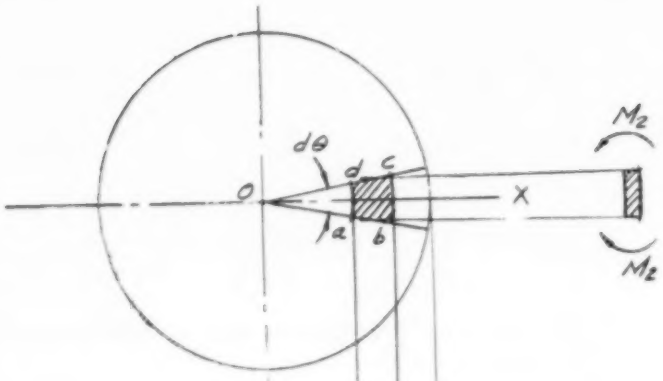


FIGURE 24D

FIGURE 24A

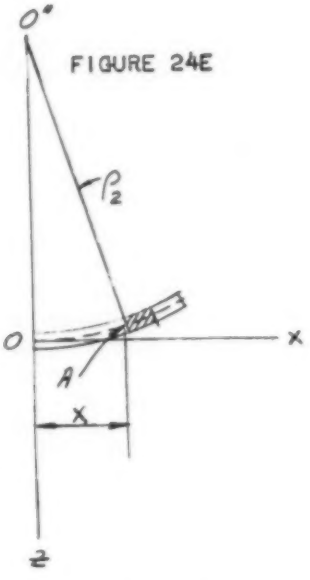
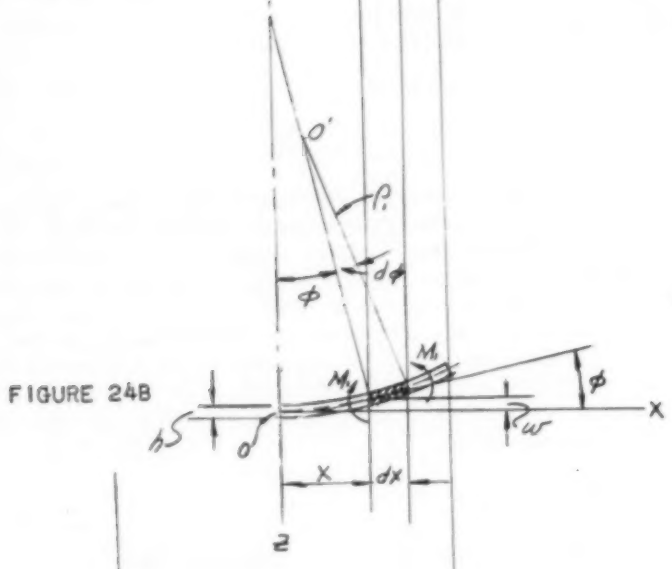


FIGURE 24E



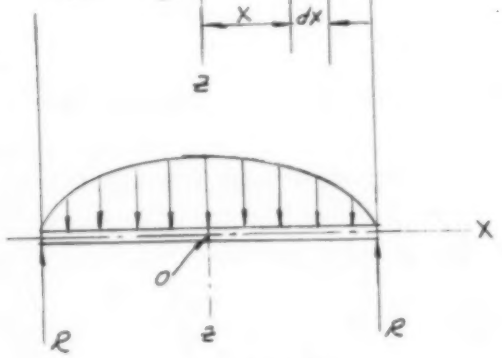


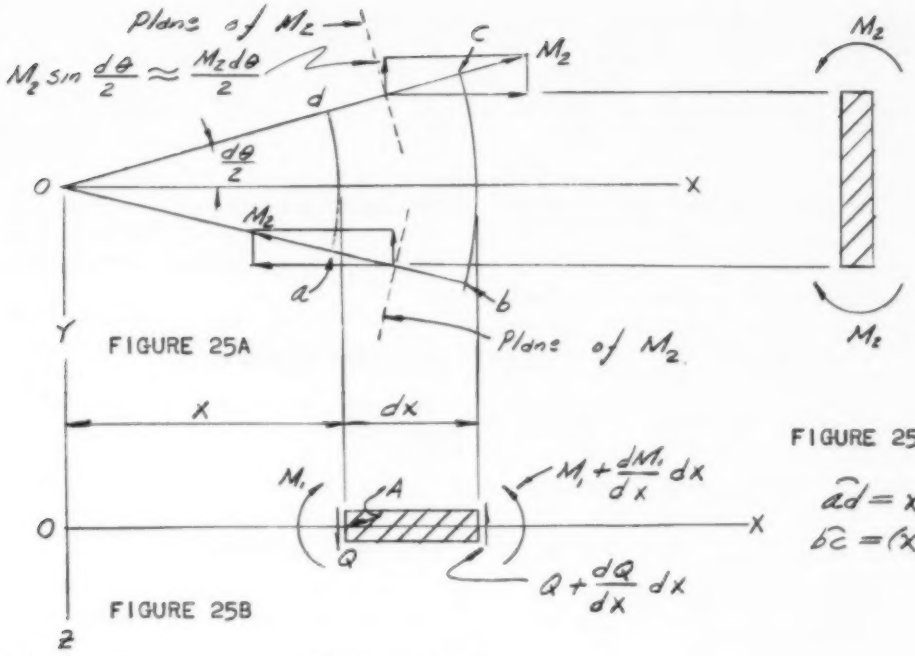
FIGURE 24C

$O'A = \rho_1$

$O''A = \rho_2$

ρ_1 = RADIUS OF CURVATURE IN THE XZ PLANE

ρ_2 = RADIUS OF CURVATURE IN CIRCUMFERENTIAL DIRECTION



Substituting in equations (37) and (38)

$$M_1 = D\left(\frac{1}{\rho_1} + \mu \frac{1}{\rho_2}\right) = D\left(\frac{d\phi}{dx} + \mu \frac{\phi}{x}\right) \qquad (41)$$

$$M_2 = D\left(\frac{1}{\rho_2} + \mu \frac{1}{\rho_1}\right) = D\left(\frac{\phi}{x} + \mu \frac{d\phi}{dx}\right) \qquad (42)$$

Taking the summation of moments about point (A) in Figure 25B

$$\Sigma M_A = \left(M_1 + \frac{dM_1}{dx} dx\right)(x + dx)d\theta - M_1 x d\theta - 2M_2 \frac{d\theta}{2} dx + \left(Q + \frac{dQ}{dx} dx\right)(x + dx)\, d\theta\, dx = 0$$

dividing by $d\theta$ and expanding

$$M_1 x + \frac{dM_1}{dx} dx + M_1 dx + \frac{dM_1}{dx}(dx)^2 - M_1 x - M_2 dx + Qx dx + x\frac{dQ}{dx}(dx)^2 + Q(dx)^2 + \frac{dQ}{dx}(dx)^3 = 0$$

Combining terms, dividing by dx, and dropping differentials of higher order,

$$M_1 + x\frac{dM_1}{dx} - M_2 + Qx = 0 \qquad (43)$$

and substituting the values of M_1 and M_2 from (41) and (42) into (43)

$$D\left(\frac{d\phi}{dx} + \mu \frac{\phi}{x}\right) + xD\left(\frac{d^2\phi}{dx^2} + \mu \frac{x\frac{d\phi}{dx} - \phi}{x^2}\right) - D\left(\frac{\phi}{x} + \mu \frac{d\phi}{dx}\right) + Qx = 0$$

OR $\frac{d\phi}{dx} + \mu\frac{\phi}{x} + x\frac{d^2\phi}{dx^2} + \mu\frac{d\phi}{dx} - \mu\frac{\phi}{x} - \frac{\phi}{x} - \mu\frac{d\phi}{dx} = -\frac{Qx}{D}$

AND $$\frac{d^2\phi}{dx^2} + \frac{1}{x}\frac{d\phi}{dx} - \frac{\phi}{x^2} = -\frac{Q}{D} \qquad (44)$$

WHERE $D = \frac{Eh^3}{12(1-\mu^2)}$ SEE EQUATION (36)

For a circular plate with a load (P) in the center

$P = 2\pi x Q$ OR $Q = \frac{P}{2\pi x}$ AND (44) BECOMES

$$\frac{d^2\phi}{dx^2} + \frac{1}{x}\frac{d\phi}{dx} - \frac{\phi}{x^2} = -\frac{P}{2\pi D}\cdot\frac{1}{x} \qquad (45)$$

For a uniform load of (q) pounds per unit area

$\pi x^2 q = 2\pi x Q$ OR $Q = \frac{qx}{2}$ AND (44) BECOMES

$$\frac{d^2\phi}{dx^2} + \frac{1}{x}\frac{d\phi}{dx} - \frac{\phi}{x^2} = -\frac{q}{2D}x \qquad (46)$$

Note for integration purposes that

$$\frac{d^2\phi}{dx} + \frac{1}{x}\frac{d\phi}{dx} - \frac{\phi}{x^2} = \frac{d}{dx}\left[\frac{1}{x}\frac{d}{dx}(\phi x)\right]$$

THE THERMAL DESIGN OF EVAPORATIVE CONDENSERS

Hugh J. Scullen[1]

Abstract

As a practical example of a method of analyzing and using heat transfer tests the data from an experimental evaporative condenser are first reduced to individual film coefficients. Then these are inserted in general equations which can be used to estimate the thermal performance of any design.

The purpose of this paper is to show how fairly elaborate mathematical methods of analysis and synthesis may very profitably be applied to test data of a relatively low degree of accuracy when the object is a comprehensive rather than precise picture of the character of a heat transfer device.

The particular kind of device used as an example is what is known as an "Evaporative Condenser." This is essentially a forced-draft, air-cooled condenser provided with means for maintaining a layer of water on the air side of its heat transfer surface so that heat will be removed from it by evaporation as well as by convection.

The heat dissipated by an ordinary dry-surface, air-cooled condenser can be computed by the well known equation

$$q = U\,A\,\frac{\Delta t_1 - \Delta t_2}{\ln \frac{\Delta t_1}{\Delta t_2}} \qquad \text{eq. 1}$$

where the factor U, called the "Overall Coefficient of Heat Transfer" is a function of the air velocity and, to a small degree, the temperature and rate of condensation.

It is common practice to show the results of air-cooled condenser tests by plotting this coefficient U against the air velocity V, neglecting the effect of temperature and rate of condensation. A much more useful way of summarizing test data is to give equations or charts for the values of the individual film coefficients which combine to form the overall coefficient. The relation between the coefficients is

$$\frac{1}{U} = \frac{1}{h_g} + \frac{1}{Rh_l} \qquad \text{eq. 2}$$

As shown in detail in Appendix 1 of this paper, the heat dissipated by a wetted-surface, air-cooled condenser can be computed in an analogous way.

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[1] This paper was presented in abridged form at the Semi-Annual A.S.M.E. meeting held in Detroit June 17, 1946 and at the first meeting of the Industrial Mathematics Society, held February 28, 1949. The data and some of the figures are used with the permission of the Nash-Kelvinator Corporation, for whom the tests were run in 1935. Hugh J. Scullen is a consulting engineer in Detroit.

REGULAR MEETINGS

The first public meeting for the purpose of organizing a society to promote the use of mathematics in industry was held on February 28, 1949. Robert Schilling, H. J. Scullen, J. E. Freer and A. W. Jacobson participated in the discussion on "Industrial Mathematics" with E. F. Riopelle as chairman. The meeting instructed the organizing committee to proceed with the preparations to formally establish the Industrial Mathematics Society. The constitution was adopted at a meeting on May 24, 1949, and the officers were elected. Dr. R. S. Burington of the Bureau of Ordnance spoke on "The Importance of Anchorage in the Application of Mathematics to Industry." After the preliminary meetings, the following four regular meetings of the Society were held during the fiscal year of 1949-1950:

October 14, 1949. Speaker: Hillel Poritsky, General Electric Company. Subject: "Some Industrial Applications of Mathematics."

December 2, 1949. Speaker: A. D. Moore, University of Michigan. Subject: "Fluid Flow Mappers."

February 14, 1950. Speaker: A. W. Jacobson, Wayne University. Subject: "Analogue Computation and Application to Engineering Problems." This meeting was followed by a two-day conference on the subject in which E. A. Caldwell of the Massachusetts Institute of Technology and a large number of local engineers took part.

May 24, 1950. Speaker: Irvin Travis, Burroughs Adding Machine Company. Subject: "Digital Computation - Its Business and Engineering Applications." Two conferences were held the next day, one on "Logic, Use and Programming" with A. W. Burks of the University of Michigan, and the other on "Design and Construction of Digital Computing Machines" with C. C. Chambers of the University of Pennsylvania.

ORIENTATION LECTURES

Three orientation lectures, arranged by the Group Activities Committee, were held during the year. These lectures by Chenea, Hay and Pajot are being published in this issue of Industrial Mathematics.

OFFICERS AND MEMBERS OF COMMITTEES
1949-1950

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Treasurer, N. C. Fisk, U.S. Rubber Company

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B. Z. Rubin, Student
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W. V. Gudaitis, Bendix Aviation Corporation

STUDY GROUPS

The following five Study Groups were active during the year:

Gear Design:

Group Leader: Douglas Hughson, Nash-Kelvinator
Adviser: W. F. Vogel, Wayne University

Computing Machines:

Group Leader: R. A. Roggenbuck, Ford Motor Company
Adviser: A. W. Jacobson, Wayne University

Servomechanisms:

Group Leader: Jorma Sarto, Chrysler Corporation
Adviser: W. H. Bixby, Wayne University

Electron Circuit Theory:

Group Leader: P. J. Willson, General Motors Corporation
Adviser: K. R. Symon, Wayne University

Stress Analysis:

Group Leader: Egon Benesi, General Motors Corporation, who also acted as an adviser.

SUGGESTIONS TO AUTHORS OF PAPERS

An author wishing to contribute a paper to "Industrial Mathematics" should submit an original and copy of his manuscript, typewritten, double-spaced on 8-1/2 x 11 in. paper. Compliance with the following suggestions will facilitate the publishing of the paper.

A. HEADING
   1. Title of paper
   2. Author or authors' names with academic degrees
   3. Present affiliation of author or authors
      (a) Position held in company (or university)
      (b) Name and address of company

B. ABSTRACT
   Include a short summary to be printed with the paper

C. REFERENCES
   1. Should appear as a separate list, at the end of the paper
   2. Form
      (a) For articles--give in the following order: authors' last names and initials, title of article, name of journal in which it appears, volume number, page number and year.
      Example: (1) Jones, J., "The Fundamentals of Physics," Journal of Physics, Vol. 153, p. 592 (1936).
      (b) For books--give in the following order: authors' last names and initials, title of book, publisher, city in which it was published, year of publication, and page reference.
      Example: (2) Janes, M. E. and Mason, E. R., "The History of Nuclear Physics," The Atomic Book Co., Inc., New York, (1948) p. 321.

D. ABBREVIATIONS
   Follow the American Tentative Standard Abbreviations for Scientific and Engineering Terms, published by the American Standards Assn.

E. FIGURES AND FIGURE CAPTIONS
   1. Photographs
      (a) Furnish glossy prints
      (b) Size should be not larger than 8 x 10 in.
      (c) Figure numbers only should be written lightly on the back of the print in pencil. Captions should be typed in order on a separate piece of paper and should contain the figure number.
   2. Drawings
      (a) Must be in India ink, preferably on tracing linen or tracing paper, but are acceptable on good quality heavy white paper.
      (b) Captions are not to be included as part of the drawing but treated as for photographs.

The editors reserve the right to make minor editorial changes in the manuscripts, so all papers may conform to the over-all style of the Journal.

The author will be supplied with 20 unbound copies of his paper, and additional copies may be obtained at the author's expense. These should be ordered when the manuscript is submitted to avoid the necessity for a rerun.